Analytic Hierarchy Process, Basic Introduction

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This tutorial uses a common situation to illustrate how to handle a multilevel decision question using a method called the AHP (Analytic Hierarchy Process), due to Thomas Saaty. The intent of the method is to produce a final, actionable list of numeric decision *priorities*, taking into account all levels of the decision structure.

I will present his method in three phases. First, a simple example illustrating the overall approach, then how to gather the judgments needed for the method, and finally, how to do the necessary calculations to produce the decision priorities. This approach is most useful when you want to prioritize various factors and/or alternatives, each within separate contexts, and then combine all of the multi-level priorities together into an overall prescription for action. That overall prescription will be in the form of a list of numbers, a priority vector, whose components conform to ratio scale measurements suitable for inter-comparisons, resource and cost benefit decisions, as well as estimation and prediction.

I have in mind specifically its use as an adjunct to an in-depth interview method targeted to knowledgeable individuals whose scaled preferences you want to capture.

Cut to the Chase -- A Party Dilemma

Let me introduce the AHP approach by teasing apart a common decision situation. Suppose you or I are planning a party for a large number of beer drinkers (substitute your favorite beverages here). What I want to do is to choose the *best beer* for the party, that's the top goal, but, there are several intermediate criteria I would like to consider before I buy. Let me restrict this intermediate set to two factors, say, *taste* and *cost-advantage*. Introducing these two factors means I have a multi-level decision hierarchy as in the diagram below. That is, I have introduced a layer of decision between the lowest level alternatives, the beers, and the top level goal, the ultimate choice. (Think of how you personally make decisions and compare what follows).

Being a lazy sort, I would usually just choose one of the alternative beers directly by intuition, or just use one criterion, like taste. AHP however, allows me to explicitly consider other intermediate and multiple criteria and then 'roll-up' or synthesize lower level decisions into an overall/top level decision. The method also has other desirable features like a documented decision trail plus a numeric measure of judgmental consistency. Using AHP, the plan is to rank the four alternative beers in terms of the overall goal, while explicitly and quantitatively taking into account the intermediate criteria.

Note: these explicit intermediate goals are always (99.5%) missing from surveys, let alone a roll-up of intermediate results, so this method really is an major improvement over the usual procedures.

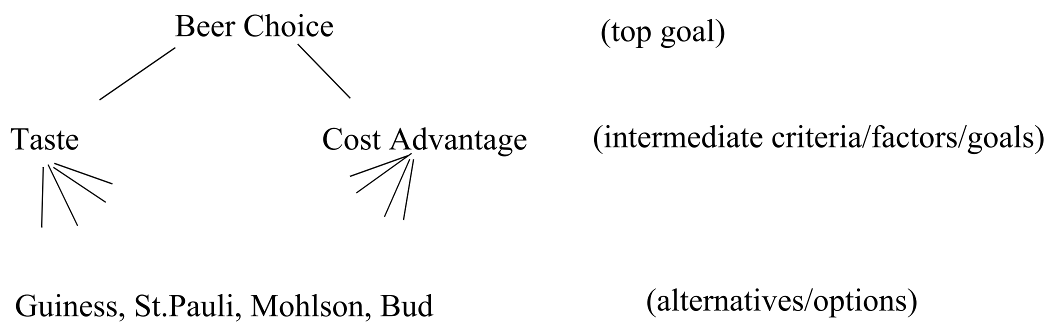

FIGURE 1. Initial Decision Hierarchy Structure - Choosing a Beer for the Party

O.k., here is the beer example. I am considering four beers - these are my alternatives/options: *Guiness Extra Stout, St. Pauli Girl, Mohlson, Budweiser.* - and my top goal, *Beer Choice*, is to choose to buy several cases of one of these for my party. What should I do? Usually, I would simply square my shoulders, set my jaw and say *taste* rules: “Guiness”, forget the cost. Let me step back a moment though, and consider another way to make this decision by including another factor, such as *cost-advantage*. While not a big deal in this situation, clearly identifying and quantifying multiple intermediate factors can make or break the presentation of a proposed course of action when big bucks or big egos are on the line.

The approach presented here, due to Thomas Saaty (see references), allows you to include both qualitative and quantitative factors in your decisions, at multiple levels. As a bonus, the method contains a way to assess the consistency of the judgements made - the only method I know of that does this. For those of you who like to see how the math you learned can be applied: linear algebra, eigen-values, and eigen-vectors will emerge as we develop the theory. For those not so much interested in the math, all these numerics can be carried out with available packages.

*As always, figuring out *what* goals, *what* criteria, *what level* to place them on, and how to structure the very questions to be asked, is really the hard part. The math plays only a supporting role, although crucial.

Cut to the Chase

O.k., back to the beer dilemma: The plan is to somehow incorporate two *intermediate* factors in the ultimate decision, *taste* and *cost-advantage*. That is, using this procedure I will get the ultimate priorities of the beer choices that implicitly contain information from the intermediate choices.

Assume the beer choices are listed as: Guiness, St. Pauli, Mohlson, Bud. Subsequent numeric lists will then refer to this order. The general flow of the method is to drill down to determine individual priorities within specific contexts and then work upwards to synthesize the individual results into an overall beer priority. You will see this below. Keep in mind that I haven’t shown you how to get these priorities, that comes later. For now, just assume it can be done so that you can follow the flow. O.k, here are the steps of the AHP, along with a brief commentary.

- First drill down and determine the rankings of the beers within the *context of taste.* That ranking will be in the form of a 4-component vector, called a priority vector. Each component represents that beer’s relative importance. For example, just suppose my judgments result in a priority (also called an impact) vector that looks like **taste** = {0.5, 0.3, 0.15, 0.05}. These are ratio values and so my preference for Guiness over St. Pauli Girl can be written as 0.5/0.3 = 1.7. As you will see later from a table that translates between numerics and natural language, the *number* 1.7 equates to a *weak* preference of Guiness over St Pauli Girl. The ratio of 0.5/ 0.15 = 3.3 equates to a *strong* preference of Guiness over Mohlson! Finally, 0.5/0.05 = 10. So I would say I prefer Guiness *extremely* over Bud. Other preferences are worked out the same way, for example, 0.3/0.15 = 2 equates to a *something between weak and strong* preference of St. Pauli to Mohlson. This illustrate the idea that numbers can express shades of meaning that are hard to say in words.

- Next, drill down and determine the priorities of the beers in the *context of cost-advantage*. Of course, the cheaper beers are going to come off looking better here. Just suppose though that my judgments determine the cost-advantage priority vector to be: **costadv** = {0.05, 0.15, 0.2, 0.6}. This says that Bud (0.6) and Guiness (0.05) are in the ration of 0.6/0.05 = 12. Quite a cost advantage for Bud! Will this advantage overwhelm my preference for taste? Hmmm, that will depend on what priorities I assign to *taste* versus *cost*, in their context, the top goal. That's next.
- Now determine the rankings of *taste* versus *cost* in their context of *overall beer choice*, that is, in the context of their goal. Perhaps this priority vector works out to be:
top= { .4, .6}. This says that I have determined, within the top most context, that the ratio of *cost-advantage* to *taste* is 0.6/0.4 = 1.5. Looks like I have prioritized cost over taste after all!

Ok, I have all I need in order to determine the ultimate priorities of beer alternatives, relative to that top most goal of *beer choice* while incorporating intermediate judgments. Looking ahead to the result, the diagram below shows that the final priority vector. The numerics place Bud as best since the *cost-advantage* weighting tipped the scales over *taste*! Gaaack!

priority of top/taste		taste/beer priorities		priority of top/costadv		costadv/beer priorities		top/final priority of beers,	
		0.5				0.05		0.23	Guiness,
0.4	*	0.3	+	**0.6**	*	0.15	=	0.21	St. Pauli,
		0.15				0.2		0.18	Mohlson,
		0.05				0.6		0.38	Bud

Note: *top/taste* means the priority of the taste factor is 0.4, in the context of the top most goal.
taste/beer means the priorities of the beers within the context of *taste*.
top/final means the priorities of the beers within the context of the top most goal (the synthesis)

Discussion of Results

In the first two steps I determined priorities of the individual beers within specific contexts, *taste* and *cost-advantage*. Then I determined the priorities of the intermediate factors/goals, *taste* and *cost-advantage* within *their* context, the top goal. Now I can combine these to finally get the overall priority of the beers relative to the top most goal. Note how the taste/beer priorities are weighted by the 0.4 priority of top/taste while the cost-advantage/beer vector is weighted by its weight of 0.6. The result is a priority vector **top/final** = {0.23, 0.21, 0.18, 0.38}. That first value, 0.23, comes from 0.4 * 0.5 + 0.6 * 0.05. The next value of 0.21 comes from 0.4 * 0.3 + 0.6 * 0.15, and so on. These numbers are based on ratio scales and so can be manipulated using standard algebra. For example, this says that Bud is 0.38/0.23 preferred over Guiness. This ratio and others, can be used for assessments and resource allocation as well as for just simple choices.

**What remains to do is to figure out how to *capture* judgments and then how to calculate *priority vectors* from these judgments. Below is Saaty's translation table. You can interpolate as well where you might judge that a "4" means something *between* weakly more important and strongly more important. Again, this illustrates the value of numerics in combination with our natural language.

GIven Element Pairs A & B, judge their relative importance/weight as below, read off the equivalent number. Use even numbers for intermediate discrimination. Note that these numbers will ultimately be interpreted as *ratios*. Row A, Column B
If A and B are equally important/weighty/better: insert --> **1**
If A is weakly more important/weighty/better that B: insert --> **3**
If A is strongly more important/weighty/better than B: insert --> **5**
If A is very strongly more important/weighty/better than B: insert --> **7**
If A is absolutely (extremely) more important/weighty/better than B: insert --> **9**

*** End Cut to the Chase ***

Gathering the data for the AHP is the most intensive part of the process since it requires multiple judgments on the part of a respondent. To make this feasible a software package (Expert Choice is a good one) can be used or you can use the approach I outline below in order to get some "feel" for the procedure before you try to automate!

Gathering Judgments Prior to Calculating Priorities

To calculate the priority vectors shown in the beer dilemma and other decision tasks, you will need to first ask yourself or other stakeholders, to make comparison judgements between pairs of factors/alternatives on a scale of 1-9 (see table above). If there are five factors, for example, you will need to ask for 10 judgment comparisons. That is, there are 10 unique pairs of judgments possible among 5 items. If four factors, then six unique pair judgment and for three factors, three comparisons. This is not hard, but it is time consuming and does suggest that you will need to aim for *quality*, not quantity in your interviews and surveys. The method only works when your respondents are knowledgeable about the topic under consideration. That is, this method is not designed for 'person in the street' questions or average perceptions, or general population feelings.

Nitty gritty suggestions for actually capturing these judgments

What I suggest is a two phase approach when asking a respondent to make judgments:

- Phase 1: First, within a given context, ask them to order the factors/alternatives by preference, with no notion yet of *how much* they prefer one over the other. This phase will allow judgements using just whole numbers with no need to figure out fractions.
- Phase 2: Now, ask them to compare all the factors, each against all the others, using the numbers 1 to 9 in the Saaty table above to quantify their judgements. This is best done in tabular form as I show below. As a rough check on the respondents understanding, their numbers should increase (or at least not decrease) as they go across a row. This is a logical consequence of phase 1.

Using the phases on the Beer Dilemma

I will use the Beer Dilemma to illustrate how to get a respondent to give you their judgments. I will pretend that the respondent is myself and I am doing an introspective process. Since there are four items, I will need to make six unique pairwise comparisons.

Phase I: In the context of "taste" what are my preferences?

Here is how I would start out. Line up the beers, in the context of *taste*, not yet saying *how much* better one is than the other but only that it *is* better. (How much better comes in phase 2).

Guiness, St. Pauli Girl, Mohlson, Budweiser.

This says that I *prefer* Guiness to St.Pauli, to Mohlson, to Budweiser, as far as *taste* goes. These comprise the table headings as shown. Notice that I haven't said by *how much* I prefer one over the other. That comes next.

Phase II: Given the preference line -up, judge the relative weights, pair by pair

Write down these Phase I preferences in tabular form as below.

Note: I only need to fill out the upper triangular values since the lower triangular numbers are mechanically determined from the upper ones. (they are the reciprocals of the upper numbers).

For example: for row one I have that Guiness compared to Guiness is unity = "1" (a logical necessity). Comparing Guiness to St. Pauli Girl, I judge Guiness to be a '3' on the Saaty scale. This is interpreted by me to mean that Guiness is *weakly* tastier than St. Pauli Girl. For Guiness versus Mohlson I give this a "4", so I would judge Guiness to be between *weakly* tastier and *strongly* tastier than Mohlson. Finally, I judge Guiness to be almost *absolutely* tastier than Budweiser by placing an "8".

Row two starts off with St Pauli versus St Pauli, logically required to be a "1". Then for St. Pauli over Mohlson I give a 2, and for St Pauli over Budweiser I give a 4. Notice that I don't need to say what St. Pauli to Guiness is since it is the reciprocal or Guiness to St. Pauli, that is, 1/3.

In row three I judge Mohlson against Budweiser and I give Mohlson a 6 to 1 taste advantage over Bud. On Saaty's scale that translates to between strongly and very strongly tastier. That's it. This table is all I need to determine the relative weights of all the beers in the context of *taste*. (Notice how the numbers must increase (or stay the same) as I go across a row. That is a consequence of lining them up as in phase 1.

Taste context	Guiness	St. Pauli Girl	Mohlson	Budweiser
Guiness	1	3	4	8
St. Pauli Girl		1	2	4
Mohlson			1	6
Budweiser				1

Mechanical consistency will require that the blanks in the table be filled by the reciprocals taken from the upper part of the table. That is, if I say Guiness over Mohlson is a "4", then I should also say that Mohlson over Guiness is 1/4, but of course, the researcher can do that later.

I have reproduced the result of a math calculation that shows the relative importance of these beers in the context of *taste*. The numbers you see are the ratios *deduced* from the paired comparisons made in the table. For example, Guiness is 0.5529/0.227 = 2.43 better tasting than St. Pauli Girl, while Mohlson is 0.171234/0.0486442 =3.52 better than Bud. The clear winner here is Guiness, which I knew going in, but now I have that Guiness is 0.553/0.049 = 11.37 better than Bud (this is off the scale but is about right)! Again, I haven't shown how these numbers are gotten, that comes in the next sections. So for now, just assume it can be done, you will learn how later.

$$\text{Priorities} \quad \begin{pmatrix} \text{Guiness} & 0.552935 \\ \text{St.Pauli} & 0.227188 \\ \text{Mohlson} & 0.171234 \\ \text{Bud} & 0.0486442 \end{pmatrix}$$

Judgments within the cost context

Notice that AHP lets you include both qualitative and quantitative judgments, all within the same process. In the *taste* context, my judgements were of a personal nature. In the case of cost-advantage I can use actual cost figures from a grocery store and insert them into my judgment matrix. This is a major advantage to be able to include both types on data, subjective and objective.

Phase I: In the context of cost what are my preferences?

Note that AHP allows you to include standard known contexts as well as strictly judgmental contexts. In this case I can explicitly look-up the *cost per ounce* of each beer and the method can smoothly include these. In the table below I have entered some (hypothetical) cost figures. To make these judgements I have taken the reciprocal of *cost per ounce* as a measure of preference and called it *cost-advantage*. For example, if Bud costs $0.13 per ounce I would convert this to a value of 1/ 0.13 = 7.69, a *good* thing! If Guiness costs $0.23 per ounce then I would convert that to 1/0.23 = 4.35. Mechanically, this would translate to a Bud advantage of 7.69/4.35 = 1.77.

Phase I Ranking the Cost Advantage of the Beers (Using the Reciprocals)

So, let me assume I have taken each cost per ounce and inverted it to get these cost-advantage values. These are my measures of 'preference' for each beer. As expected, Bud does very well in this category.

- Bud = 7.69
- Mohlson = 6.67
- St Pauli Girl = 5
- Guiness = 4.35

Phase II: The paired judgments can be mechanically filled in.

Now the table below can be mechanically filled out since there is no judgment involved since all the costs and their reciprocals are known. The reader probably can see that I wouldn't need to construct this table in order to compute the relative weights of these beers, but seeing how the ratios are entered might give some insight into the process underlying other qualitative judgments.

Cost context	Budweiser	Mohlson	St. Pauli Girl	Guiness
Budweiser	7.69/7.69 = 1	7.69/6.67	7.69/5	7.69/4.35
Mohlson		6.67/6.67	6.67/5	6.67/4.35
St. Pauli Girl			5/5	5/4.35
Guiness				4.35/4.35

After some calculations, the priorities (relative cost advantages) of the 4 beers are:

$$\text{Priorities} \begin{pmatrix} \text{Bud} & 0.324477 \\ \text{Mohlson} & 0.281213 \\ \text{St. Pauli} & 0.21091 \\ \text{Guiness} & 0.1834 \end{pmatrix}$$

As expected, Bud is the best in this category in ratios of 0.32/.28 =1.14 with respect to Mohlson and 0.32/.18 = 1.8 with respect to Guiness.

Ok, now what?

We now have the alternatives prioritized against each intermediate criterion, *taste* and *cost*. But what is the relative importance of *taste* to *cost*? Remember, a comparison can only be made within a given context. In this case the comparison is within the top most context. This will be very simple, so stay with me!.

Top goal= Beer Choice	Taste	Cost Advantage
Taste	1	1/3
Cost Advantage	3	1

So, for my party where I expect a lot of guests, *cost advantage* is (weakly) more important than taste, not overwhelmingly, but still more important. Calculating the relative weights of these two factors in the context of the top goal yields the priority vector below. Since I am having a lot of guests, cost advantage becomes important, even a little more important than taste.

$$\text{Priorities} \begin{pmatrix} \text{Taste} & 0.25 \\ \text{Cost} & 0.75 \end{pmatrix}$$

Rolling Up the intermediate judgements into the top goal context.

We now have the picture below where I have placed the priority values on the links in part (a). Once I have all these priorities I can combine them as in part (b) to get the overall priority vector (c) showing the relative importance of the alternatives. No real surprises here since the initial *taste* judgement was so overwhelming in favor of Guiness! But is does show the relative improvement of Bud versus Guiness due to the importance of *cost advantage* over *taste*. The reader will note that this doesn't match the initial beer dilemma example I gave, but this one is a bit more realistic!

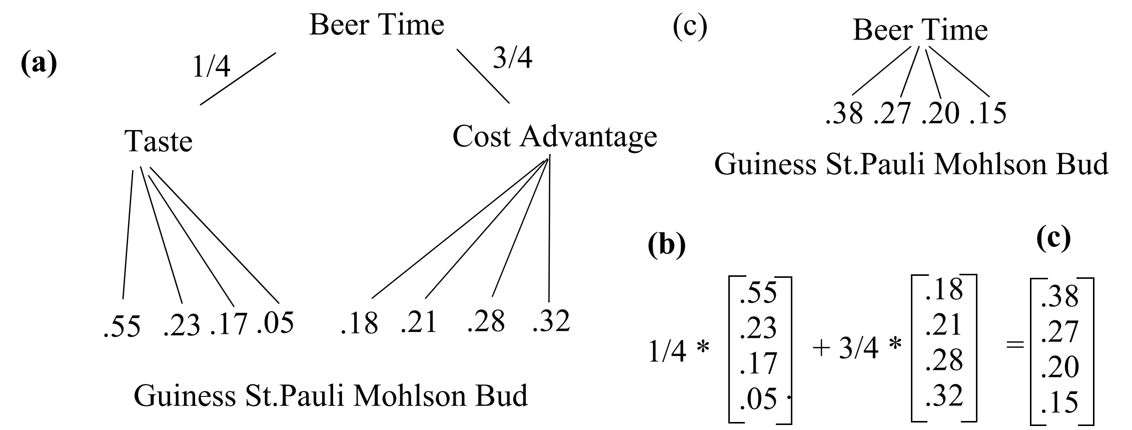

FIGURE 2. Weighted Decision Structure - Choosing a Beer for the Party

Intuitive Features of AHP

You will need to explain to the people what they are supposed to do when they use this AHP process and how they can do a little self checking as they go along. Remember, once a person has ordered the factors from highest preference to lowest, the numbers in a judgement row can't go down, they must go up or stay equal. Here is another weights example where I show what the judgments would ideally look like.

Another weighty example

This next example shows you why AHP works and gives some insight about how it works. In this made up example we, the observers, know the actual weights of four objects while a respondent Kim, can only pick them up in pairs and make a judgment. Using the example of actual 'weights' matches the underlying ideas behind AHP. The method must reduce to actual weights when you have complete information.

Phase I

Suppose a respondent, Kim, has ranked some boxes labeled A, B, C, D in the context of their Weight. (Kim doesn't know the box weights but we do!) as below: That is, before Kim does a detailed examination, she has judged these weights in the order, most important (weighty) first as: B, A, C, and then D.

This is common in surveys: you are asked to rank, say, four factors in importance using numbers 1 through 4. Same thing here, except I am asking Kim to write the labels rather than numbers in order. O.k, here is Kim's line-up. B is most important (weighty) followed by A then C then D. This doesn't say how *much more* important one is than the other nor the overall importance priorities, nor anything about judgemental consistency. That's phase II.

Phase II

B	A	C	D	// Kim has written these first, based on a *Weight* criterion
100 lb,	80lb,	50lb,	10lb	// note that Kim can't see these values, but we can
1	1.25	2	10	
	1	.1.6	8	

1 5

Now, for the second phase, **If Kim had perfect judgement** she would say the following:

For the first row, she would judge the importance of B over B as “1”, B over A as 100/80 = 1.25, B over C as 100/50=2 and B over D as 100/10=10. Notice that the numbers in a row can only go **up (**given that we have previously ranked them in importance).

For the second row, she would start with column “A” and judge A over A as “1”, A over C as 80/50= 1.6, and A over D as 8.

For row three she has, C over C= “1”, C over D = 5.

Done! She needed 6 comparisons.

Now, your task as a researcher begins.

You now fill out the complete matrix by taking reciprocals to get the matrix shown. Notice that if the respondent had perfect judgement, they could have placed these *actual weights* as numerators and denominators, as shown!

```
matrix = {
   {100/100, 100/80, 100/50, 100/10},
   {80/100, 80/80, 80/50, 80/10},
   {50/100, 50/80, 50/50, 50/10},
   {10/100, 10/80, 10/50, 10/10}};
```

Doing some math calculations (explained in the AHP main tutorial), Kim gets the priority vector of boxes with respect to weight, as follows:

$$\text{Priorities} \quad \begin{pmatrix} B & 0.416667 \\ A & 0.333333 \\ C & 0.208333 \\ D & 0.0416667 \end{pmatrix}$$

As expected, for example, the ratio of B to A is 0.416667/0.333333 = 100/80, B to C is 100/50, and so on through the remaining priorities. So, given accurate judgments, this approach yields the relative importance of these factors (in the context of weight). Notice also that I haven’t calculated the consistency ratio since these are perfect judgements, pretty unusual I would say! All the details are below and as you can see, the consistency ratio is zero.

```
Eigenvector  = {10., 8., 5., 1.}

Eigenvalue  = 4.

             ⎛ B 0.416667  ⎞
Priorities   ⎜ A 0.333333  ⎟
             ⎜ C 0.208333  ⎟
             ⎝ D 0.0416667 ⎠

Consistency  Index  = 0.

Random  Consistency  Index  = 0.9

Consistency  Ratio  = 0.

**Note : A Consistency  Ratio  of
  less  that  0.10  is considered  acceptable
```

Calculating any Priority Vector

Here is a rough and ready way to calculate *any* priority vector, given that you have filled out the paired comparisons table.(Adapted from Saaty's examples)

1. *Normalize* each column, that is:
    1. Sum the values in each column
    1. Divide each element in that column by its sum. This causes the total of a column to be one, and so is called Normalized. Do that for every column.
2. Sum each row and divide by the number of columns.
3. The resulting vector is (an estimate of) the priority vector. The numbers in this vector are ratios and allow you to assess the ratio importance of the associated factors.
4. Call this resultant vector the priority vector and denote is as **p**, it will be used in the next step.

Calculating the Eigenvalue

1. *Matrix multiply* your paired comparisons matrix with the priority vector **p,** call this result the vector **V**.
2. Divide **V** by **p**, component by component.
3. Average the components of the result of step 2, that is the approximate *eigenvalue*
4. The eigenvalue is just a number, like 3.28
5. Call this resultant value the eigenvalue and denote it by **ev.**

Calculating How Consistent Your Judgments Are

1. From the eigenvalue step above, calculate the consistency index = CI as (**ev** -1)/(number of columns -1)
2. Divide CI by the random consistency value from the table "Random Consistency Matrix Values"