Basic Math/Basic Physics (* Draft 2011-03-06)

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Exploring Motion and Math (Distance = Rate * Time)

Pondering on my physics book (Halliday), I just realized what the author had been saying all along about the possibility of the breakup of motion corresponding to various causes. That led me to look at the simplest equations of motion in hopes of visualizing what led to such an equation as:

Distance traveled = initial velocity * time + 1/2 * acceleration * time^2

By drawing some pictures I hoped to convince myself that this equation was understandable enough to be able to use it in further investigations. Maybe the drawing below will enable you to do the same (unless you already have this down cold)! Just jump in and all will become clear. [rob r.]. Note: An addendum to this note discusses *Harmonic averages* that are useful to find *average rates*.

Visualizing Distance in terms of Time, Velocity, and Acceleration

Suppose I'm driving my Ferrari Targa up the on-ramp heading for a freeway, and initially moving at a steady 40 mph. Suppose I want to accelerate and get up to 60 mph as I enter freeway traffic. If I accelerate at 5 mph per second, it will take me 4 seconds to get up to 60 mph. How far have I traveled during this 4 seconds? That amounts to calculating a distance over a time where the velocity is *not* constant and that's what I wanted to better understand. See the diagram below for a picture of this. I will use these numbers to show how to think about the calculation of the distance covered during this on-ramp trip. Solving this puzzle will reveal an insight into the calculation of other equations of motion and even an insight into Kinetic Energy (energy of motion).

The take home idea is that my initial 40 mph can be treated as a constant, separate but simultaneous, to the accelerated motion that increases the 40 mph up to a final velocity of 60 mph. That separation, that decomposition into different causes, is the key to understanding the connection between distance covered, time, velocity, and acceleration.

I will use English units of miles, hours, feet, and seconds. The text in square brackets will be the units the variables are expressed in,

Variables and their units

v0 is the initial velocity (I will express this in miles per hour, written as: 40 [mi/hr]

v is the final velocity: 60 [mi/hr] (this is the velocity at the end of the acceleration)

vbar means the average velocity = (v0 + v)/2 = 50 [mi/hr]

a is the acceleration expressed as change in miles per hour, per second. This is actually a common way to express acceleration here in the U.S: Just think of the phrase: "from zero to 60 in 4 seconds!". This means that for every second, the miles per hour increases by 60/4 = 15 [mi/hr], So that at the end of 4 seconds, I am doing 60 mph, that is, 60 [mi/hr]. (Assuming I started from zero). If I had instead started at 40[mi/hr] and accelerated at 5 [mi/hr/sec] for 4 seconds, I would again reach

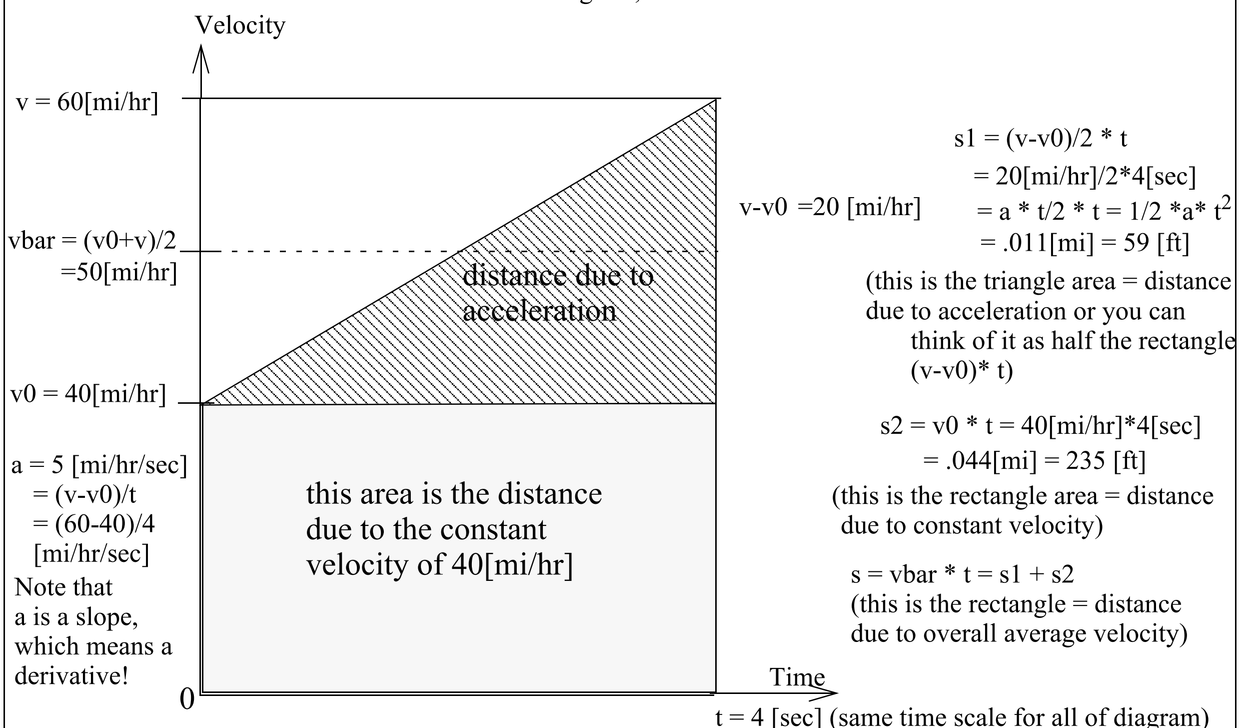
60 mph but be much less impressive.

a = (v - v0) /t [miles per hour per second] (written as [mi/hr/sec]) the 'per' indicates the unit goes in the denominator. This is actually the definition for constant acceleration. For this example, since I have already assumed a starting and ending velocity, I can calculate acceleration as:

60 [mi/hr] - 40[mi/hr])/4[sec] = 5 [mi/hr/sec], which just means that the Ferrari is gaining 5 mph every second. That is a 'rate'. In the next figure I am going to show how all the quantities can be represented as products of a 'rate' times a time. That product will be the area of a rectangle that you can identify from the picture.

Exploring Distance = Rate * Time
(the diagram below is a velocity versus time plot)

Note: the assumed values below on the left of the diagram, 'a' is constant



Equivalent Expressions for the Distance (S) as a function of Time, Velocity, and Acceleration

S = vbar * t = (v0 + v)/2 * t (this is the overall rectangle representing distance covered)

S = v0 * t + 1/2* {v-v0} * t (this expresses S in terms of two rectangles

S = v0 * t + 1/2 * {a * t} * t (note that {v - v0} = a*t)

$S = v0 * t + 1/2\ a * t^2$

as a bonus, since t = (v-v0)/a, replace 't' in the last equation to get:

$v^2 - v0^2 = 2\ a * s$ (this connection is critical in expressing Kinetic Energy = $1/2\ m\ v^2$)

FIGURE 1. Relating distance (s), to time (t), velocity (v), and acceleration (a).

Notice that Figure 1 expresses the velocity in terms of time, v[t] = v0 + a * t; This leads to the next diagram showing that the areas in figure 1 can also be thought of as integrals.

$$s1 = \int_0^4 v0\, dt = v0 * t$$ (recall that the area of s1 was v0 * t)

$$s2 = \int_0^4 a * t\, dt = 1/2\, a * t\^{}2$$ (remember that a = (v-v0) * t, so this is s2)

$$s3 = \int_0^4 vbar\, dt = (v + v0)/2 * t$$ (this comes from (v+v0)/2 * t and is the overall average velocity, so this is s)

FIGURE 2. Expressing the areas of Figure 1 as integrals

Finally, here is a picture of the distance directly as a function of time. It is actually a parabola but doesn't bend much due to the small value of the acceleration term.

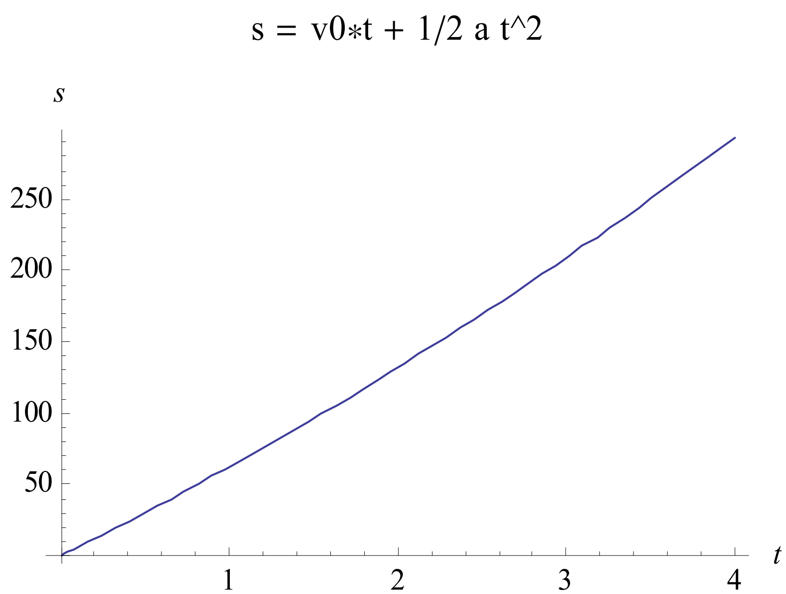


FIGURE 3. s = v0*t + 1/2 a * t^2 Graph of distance as a function of time

Yes, I See Where it is Now (at t = 0), but Where Was It Earlier?

So far, I have simply written the distance equation as depending on the initial velocity v_0, and a constant acceleration, 'a', as below, where 's' is the distance from an origin and depends on the time of travel. So I write s[t], meaning that 's' depends on 't':

$s[t] = v_0 * t + 1/2\, a * t^2$

Notice that this equation assumes that when time = 0, I am starting at the origin. Suppose though, I am starting out at a distance of s_0 from the origin when I start my stopwatch, How do I interpret the equation now?

So, let me change the above equation just a little to take this *enrichment into account and consider that I start off initially at some distance from my origin, say at s_0 feet. The general equation of distance now looks like:

$s[t] = s_0 + v0* t + 1/2\ a * t^2$

Let's get specific using the values of the starting "Ferrari" example

v_0 = 40 [mi/hr] = 58.7 [ft/sec]

a = 5 [mi/hr/sec] = 7.33[ft/sec]

s_0 = 100 [ft] (this is the new component of the equation and is the initial displacement from the origin)

The distance equation now appears as:

$s[t] = 100[ft] + 58.7[ft/sec] * t[sec] + 1/2* 7.33[ft/sec/sec]] * t[sec]^2$

At what time will the distance be zero? Solving this quadratic equation will naturally yield two times that satisfy the equation, but what do those times mean?

The roots of this equation are:

t = -1.94 [sec] and t= -14.1[sec

Plot of the New Distance Equation.

Let me simply show a time versus distance picture of the s[t] equation, as below. The equation roots are at about -2 seconds and -14 seconds. The math doesn't care what those numbers mean, only that they satisfy the equation. (Think of math as a "domain independent world view").

My interpretation is that my car starts at the origin at t= -2 seconds. Even though I started my watch at t=0, the car had started traveling earlier and had passed through the origin, s==0, at t = -2 seconds. The question remains though, when did it initially start moving? Here is where my physical knowledge kicks in: The car started moving for the very first time when the velocity was zero and increased from there. To find *that* time I can look at my diagram below and pick out the time where the curve bottoms out, which is at t = -8 seconds, that's when velocity is zero, that's when the car started.

So, my take on this equation is: my car initially started out 8 seconds (t= -8) before I started my stopwatch, passed through the origin (s==0) at t = -2 and when I actually started my stopwatch (t=0), it was 100 feet beyond the origin.

Question: What do you make of the fact that there is another time at which the car (virtual?) passes through the origin?

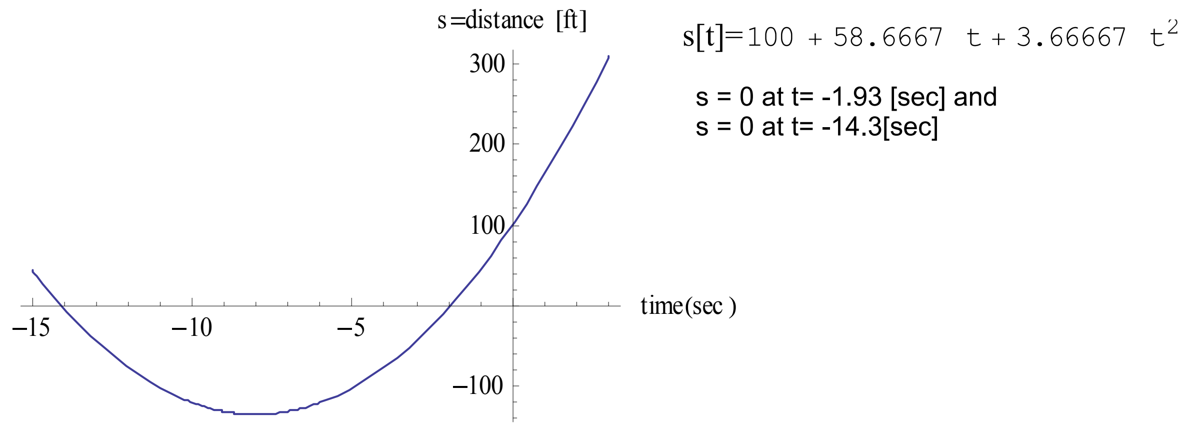


FIGURE 4. Distance equation that starts the stopwatch (t=0) when car has already gone100 feet.

Motion Summary

Assuming constant acceleration, such as the example above or, say, the constant acceleration due to gravity, makes the equations of motion fairly easy to write down and to visualize. I have made a start here in describing the simpler equations of one dimensional motion. They boil down to an application of the basic idea that *distance = rate * time.* As an aside, the equation in the last line in Figure 1 on page 2 is very important in further studies of energy: $v^2 - v0^2 = 2\ a\ s$

when multiplied by mass (m) and slightly re-arranged, it shows

$1/2 *m * v^2 - 1/2 *m *v0^2 = m * a * s = m * \{(v - v0)/t\} * \{(v + v0)/2 * t \}$

The difference on the left is the change in Kinetic Energy, while the right hand side is Newton's (net) force (m * a) through a distance expression (s). Force in the direction of motion for a given distance is work, which equals the change in Kinetic Energy.This is the *Work - Kinetic Energy* Theorem.

Engineering/Physics use of Dimensional Analysis as an Important Advantage Over Pure Math

In the last section I point out a subtle feature of math equations, *you* need to interpret them! In that equation I talked about where the Ferrari was at *negative times* since I started the stopwatch, time = 0) when the car had already traveled 100 feet! Notice that putting the equations in the context of everyday experience helps a lot, especially including the *dimensions* of the equations. Dimensions tell me if I go wrong. That is, if the left side of the equation is in terms of *feet*, then every term on the right side must reduce to a dimension of *feet*. That's one more advantage over a pure math manipulation which doesn't have attached dimensions.

Other Averages - Harmonic Means

Often you will need to calculate *average rates* over a given *distance* rather than average rates over a given time. That is, suppose you want to calculate an average rate for two cars that travel a distance of 120 miles at respective rates of 60 mph and 40 mph. Your first impulse to say, (60 + 40)/ 2 = 50 mph, turns out to be wrong! That would work if they both traveled for the same amount of *time*, but that's not the case here.

However, your basic idea *rate = distance/time* still underlies the following analysis. The only difference is that now I will replace time by *average time*. Ok, here goes (note that *distance* is the fixed quantity in this derivation)-

average rate (RBAR)= distance /(*average* time to travel that distance)

let t1 (time) = distance/(rate of car 1) = 120 / 60 == 2 hours

let t2 (time) = distance/(rate of car 2) = 120/40== 3 hours

average time = (t1 + t2)/2 = (120/40+ 120/60) /2 = 2.5 hours

average rate (rbar)= 120 mi /[(120/60 + 120/40)/2]= 1/[(1/60 + 1/40)/2]= 48 mph

If you go on to study electronic circuits you will see this same idea when you calculate the average resistance of two resistors, R1, R2, in *parallel*. The analogy becomes that Resistance is the *Rate of Dissipation* of energy across the resistor leads. So Resistance takes on the role of a Rate. In this case the common factor is the same *voltage* (V) across both resistors. The analog becomes: Voltage == Distance, Current ==Time, and Resistance == Rate.

Now the average can be computed as:

average resistance = voltage/(average current through the resistors)

Suppose i1 is the actual current through R1, while i2 is the actual current through R2.

average resistance = voltage / (average current)

average current = (i1 + i2)/2 = (V/R1 + V/R2)/2

average resistance (Rbar) = V/ ((V/R1 + V/R2)/2)

Rbar = 2 / (1/R1 + 1/R2)

This technique works for any number of resistors in parallel. So, if I had three resistors in parallel, I would get an average resistance:

Rbar = 3 / (1/R1 + 1/R2 + 1/R2)

Torques about Axes and Torques about Planes

Chap 9 of Halliday, *Systems of Particles*, talks about finding the *coordinates* of the center of mass (CM). That is where all the mass may be considered to be concentrated. The CM motion is in turn determined by the (vector) summation of all external forces applied at that (virtual) point. That is, there may not be actual mass at that point as in a donut or a horseshoe. The coordinates of the center of mass, in 3D, may be calculated by considering the torque around each axis in turn. What I finally realized is that in 3D, the torques along an axis can be considered as turning a *plane*, with the normal to the plane being parallel to the axis along which the masses are hung. (I am using gravity as the intuitive turning force here, taking each axis in turn).
For example,in the diagram “Torque due to hanging masses in a direction parallel to X axis” on page 7, the coordinates of the center of mass along the X axis depends on the distance, in the X direction, out from the Y-Z plane and the mass amount hung there! This displacement of the mass out from the Y-Z plane represents a turning motion imparted to the Y-Z plane. (assuming the active force here is gravity with masses hanging *down* from axes parallel ot the X axis. In that diagram, two masses are hanging along the X axis, *m1* and *m2.* A third mass, *m3*, is hanging out from the Y-Z plane, at a distance which is parallel to the X-axis. That is, any distance out perpendicular from the Y-Z plane, puts me parallel to the X axis. I indicate this direction by a normal vector to the plane, *n*. So, a mass anywhere out from the Y-Z plane, can be considered as hanging down from a line (parallel to the X axis) projecting out from that point. All of the torques due to *m1*, *m2,* and *m3* then determine the X coordinate of the center of mass.

To determine the Y coordinate of the center of mass, I consider the same masses now hanging down from the Y axis or an axis parallel to the Y axis some distance out from the X-Z plane.

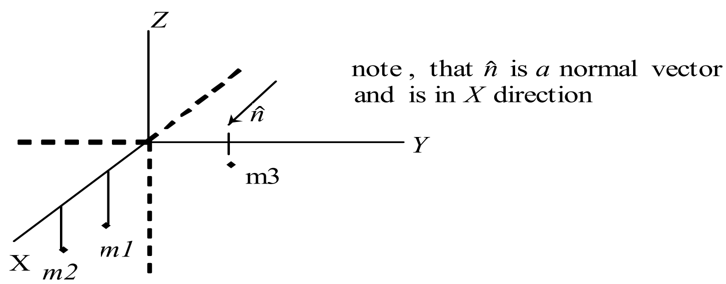


FIGURE 5. Torque due to hanging masses in a direction parallel to X axis

2-D Motion [2011-03-06]

(Didn't take the time to see where or if I had covered this, but was doodling with eqns from the *Joy of Physics* book and was playing with the eqns of motion again) Had a mini-micro realization about one of the factors in the derivation!

By definition, starting out I know that (x - x0)/t = vbar, so not surprisingly, that turns up in the derivations although Ididn't see it early on.

x - x0 = v0 *t + 1/2 a t^2, factoring out the 't' I get

x - x0 = t (v0 + 1/2 a t) , but the last factor is actualy *vbar*!

(x - x0)/t = (v0 + 1/2 a t) // back to square zero since this RHS is vbar

Since:

v0 + 1/2 a t = v0 + 1/2 (v-v0), since *a t* IS v-v0 for constant *'a'*

and finally, v0 + 1/2 (v -v0) == 1/2(v + v0) == vbar

2-D Motion, finding the range of a parabolic trajectory

Continuing the doodling, if x0 = 0 and only gravity is at work, then a particle launched at an angle *theta* follows a parabolic path. Time is represented by the variable 't'. The *range* is the *x* value where the particle hits the ground again. It turns out that the equation describing this path is a parabola and is a quadratic in x, that is, it has two solutions since x appears to the second power. One solution, as we will see derived below, is that x== 0 is a possible range solution, but not too interesting). The second solution shows that the range of travel of the particle, that is the x distance traveled before it hits the ground again, depend on the launch velocity and the launch angle, just as you would expect.

O.k., here is the derivation: note a curious but crucial fact of velocity, acceleration and forces -- then can be resolved into independent components along orthogonal axes. This makes solving a complicated 2-dimensional equation, equivalent to solving two 1-dimensional equations, as you will see below. “Divide and conquer”!

Let vx = v0 Cos[theta] be the initial X direction velocity. and vy = v0 Sin[theta] be the initial Y vertical launch velocity. *theta* is the launch angle. v0 is taken to be the speed of the launch (absolute value of the vector velocity) that can be broken down into two components, one along the X axis and the other along the Y axis, this is vx and vy.

(Up is plus and down is negative along the Y-axis which means that the direction of the gravity force is taken as *down.* Right is

). Then the two equations are:

[1] x = vx * t // for the horizontal distance traveled in time ‘t’ (it’s independent of the y motion)

[2] y = vy * t - 1/2 * g* t^2 // the minus is because the gravity force is taken as the minus direction.

then, solving for the ‘t’ in equation [1], and substituting into equation [2], I get:

[3] y = vy * x/vx - 1/2 g (x/vx)^2 // this is a quadratic in x, which is a parabola in this case

To make this equation easier to work with, factor it into 2 components as in [4]

[4] y = x/vx *(vy - 1/2 * g * x/vx)// two factors here, so setting y==0 means two equations to solve

The *range* is the *corresponding x* value such that y is back at ground level, that is, y ==0, so, set y== 0 and solve the quadratic [4]. Note, I have factored it so it is a bit easier to solve since y==0 means that each factor could be zero. So this allows me to solve two easy equations, one equation for each factor.

One factor says that y==0 when x/vx ==0 which means I didn’t move at all, and y is still zero. The solution is simply, x ==0.

The second factor, when set to zero, has the interesting solution :

vy - 1/2 *g * x/vx ==0

x = 2 * vx * vy /g

x = v0 * v0 * 2* Cos[theta] * Sin[theta]/g

// there is a trig indentity that says that 2*Cos[theta]*Sin[theta] == Sin[2 theta], there is a derivation of this in another tutorial on the milagrosoft.com website, *TrigBasics.pdf.*

So finally, we can see that range depends on the square of the launch speed, as well as the angle. The maximum range is when Sin[2 theta] ==1. This happens when theta = 45 degrees.