Exploring Growth with Algae [**draft 2009-09-27**]

A side excursion showing the relation of growth, Present and Future values, to exponentials and logs.[rob rucker 2009]

Let me take a little time and work through a few examples of how you might understand a fundamental growth process that occurs throughout the natural world and our adaptation of it, particularly as it concerns compound interest of money. Along the way we will learn two fundamental functions that together express this growth: the exponential and the logarithm. I will develop this discussion in discrete steps so you can follow along easily

An Algae Example

From the organic world here's an example I have used for my (engineering) economics classes.

Suppose I want to interest some investors in developing an algae farm to supply vitamin components. Although I've never grown algae before, I'm pretty good with numbers, and from the available data, I can calculate projected yields that ought to make potential investors very eager. Using available data, I find that really fast growing blue-green algae can potentially grow at an astounding rate of 100% per month, given a special nutrient diet and adequate space. Looks to me like a few swimming pools and some algae food is all I need to retire early!

So, I propose setting up several 'algae ponds' that I grow my algae in and then periodically harvest and ship. I will sell my algae 'crop' by the square foot since the 'mat' depth stays relatively constant, according to the data, due to the limited depth of sunlight penetration. The algae usually grow in a solid expanding circular pattern that I can start from an initial 'seed' circular mat. Note: A solid clump of algae is often called an algae 'mat'.

Projected Growth Calculations for my Algae Farm Business Plan (Future Area)

What I am doing here is calculating the future 'area' under a given growth rate assumption. (In the finance world this is called the Future Value (FV) of an initial 'mat' of money). For my projected calculations, I will assume that my algae are the very active type that routinely expand in area by a proportional factor of 100% of their current area, per month. (This assumes that I feed them those special nutrients they need and that they keep reproducing). This explosive growth pattern is a common feature of fast breeding organisms such as bacteria, algae, or even those famous *Fibonnaci* rabbits! Organisms like these often increase in numbers as a *proportion* of their *current* number, or in my algae case, their current area. To impress my potential investors I lay out the following scenario that I will describe in terms of month units of time:

I start with 1 square foot of algae at the start of *month 1*. (This 1 square foot would equate to an area of a circular mat with a 13.6 inch diameter). If the algae continue to grow at the 100% rate, then at the end of that first month I calculate I will have (1 + 1.0 * 1)= 2.0 square feet of algae.

> Note that you are used to seeing growth (interest) rates down in the lower percentages, like 8% or 23%. The decimal equivalents of those would be 0.08 and 0.23. In the case of my algae's 100% growth rate, the decimal equivalent is 1.0. I will explain how to find areas due to growth rates other than 100%, at the end of this section.

So, I start month 2 with a mat area of 2.0 ft^2. Since the second month starts off with that 2.0 square feet and increases at the steady rate of 100%, at the end of the second month I calculate a mat size of: 2.0 + 1.0* 2.0= 4.0 square feet of algae. (As an aside, if I can get $50 per square feet of algae, then I just made 4.0 * $50 =$200 just for watching algae grow for two months. This should really please investors).

Below is a diagram showing how the algae grow, detailed calculations follow.

Algae growing at a @100% rate/month, starting with 1 square foot of algae, over 5 months
(Note: *end* of month areas calculated based on area at *start* of that month)

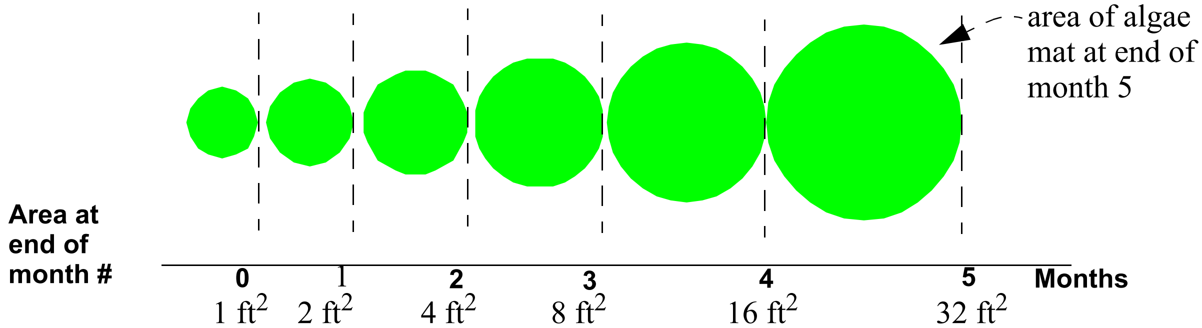

FIGURE 1. My Projected Algae growth pattern: month 0 (‘initial ’), through month 5

To translate this to some math that we will use later, let me use the letter ‘A’ to stand for area and A[month #] to mean the area at the *end* of a month numbered 0, 1, 2, 3, 4, or 5. For example, A[month 0] is the area at the end of month 0 (this is just a convenient way to indicate the initial starting mat of size 1 ft^2). A[month 1] would mean the area of the algae ‘mat’ at the end of month 1, which would be 2.0 ft^2. A[month 2], A[month 3],A[month 4], and A[month 5] are interpreted similarly as in the above diagram. At the end of month 5 I will harvest and ship.

Math Detail for the Area Calculations, (assuming 100% growth rate)

Area at the end of month 0 = 1 ft^2. (Initial size of starting algae mat)

Area at end of month 1= 2.0 ft^2.(calculations below)

A[end of month 1] = A[end of month 0] + 100% * A[end of month 0] //note: 100% equates to 1.00

= A[end of month 0] * (1+ 100%) = A[end of month 0] * (2.0)

= 1 * (1+ 1.0) = 1 * (2.0) = 2.0 ft^2

Area at end of month 2

A[month 2] =A[month 1] + 100% * A[month 1]

= A[month 1] (2.0) = (A [month 0] *(2.0)) * (2.0) = 1 * $(2.0)^2$

= 4.0 ft^2.

Areas at the end of each month:

A[month 0] = 1 ft^2 note that I also could write this as 1 * $(2.0)^0$ since $(2.0)^0 =1$

A[month 1] =1* $(2.0)^1$ = 2.0 ft^2

A[month 2] = 1 * $(2.0)^2$ = 4.0 ft^2

A[month 3] = 1 * $(2.0)^3$ =8.0 ft^2

A[month4] = 1 * $(2.0)^4$=16.0 ft^2

A[month 5] = 1 * $(2.0)^5$ =32.0 ft^2

Turns out though, as great as these numbers are for my bottom line, *the reality is even better*!

The Value of Actual Experience - Actual Growth Patterns

Well, it's 5 months later, the investors bought into my plan and now I have started growing my algae. I set up a pond to do some baseline production data, and started with a mat of 1 ft^2. Curious things happened. That first month, the area was bigger than I calculated and each month thereafter the area was a lot bigger than I had calculated. At the end of month 5, the growth was astounding. Instead of the 32 ft^2 I had calculated, I measured a mat of area a little bigger than 148 ft^2 ! This works out to a mat with at 13.6 ft. diameter.

Whoa, what happened here? Could my math be that poor? Yep. Can you see the assumption I made that is not quite right? If you picked up on the phrase "proportional to the current area" you are on the right track. What that means is proportional to the current area, at *every instant of time*.

The mistake I made was to calculate the months' growth based on the area at the *beginning* of each month, not taking into account that every second past that beginning, the area is also increasing and so, the proportional amount being added is also increasing, based on that instant. The algae are actually growing continuously during that whole month, but I only took into consideration their area at month beginning times. Lets see how to give a more honest accounting.

Looking at the very first month of growth, breaking it into 2 periods and then 5

Let me go more deeply into the idea of continuous growth by approximating it by breaking up that *first* month of growth into two periods. We already know that the algae grow continuously but initially recall that I calculated the growth that first month by just using the initial size of the mat (1 ft^2), and bumping it by 100%. to yield a final size for that first month of 2.0 ft^2 . This is not fair to the algae though, since they were growing continuously throughout that first month. So, let me break that first month up into *two* periods such that during the first half I calculate their growth based on 50% (reaching a size of 1.5 ft^2) and their growth for the second half of the month at 50% of that intermediate size, reaching a week ending size of: 1* (1.5) *(1.5) = 2.25 ft^2. Notice how this is already (slightly) larger than the calculation based on just using the size at the end of the previous month. Recall that A[month 1] was 2.0 in the previous scenario.

The Detailed calculations for breaking that first month into 2 periods

The first half of the month, based on 50% (100%/2) growth rate, yields an area of 1* (1 + 1.0/2) = 1.50 ft^2. Now, that is the size going forward for the second half of the month. The second half of the month's growth starts out with 1.50 and bumps that by 50% to yield a total given by: 1.5 * (1 + 0.5)=2.25.

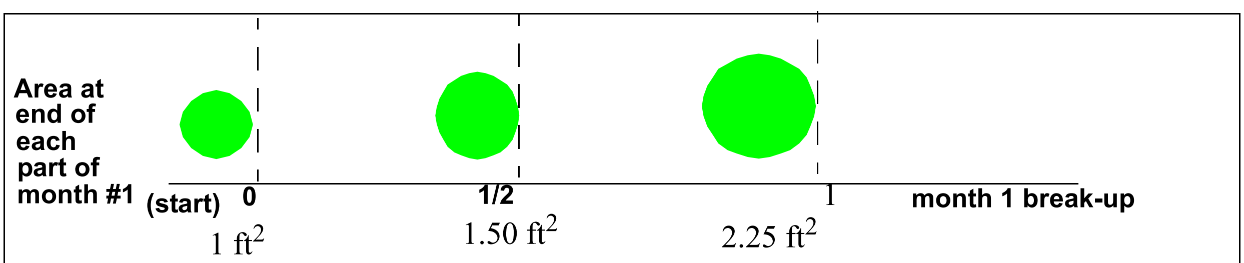

FIGURE 2. Break up first month into two equal periods, at 100%/2 = 50% per period

Notice that I can compactly find this final size by calculating:
$1^* (1 + 1.0/2)^2 = 1^* (1.5)^2 = 2.25\ ft^2$

(remember that the first "1" in the above equation is the starting size mat of 1 ft^2)

Breaking that first month into 5 periods, with fractional growth rates per period of 100%/5 = 20%

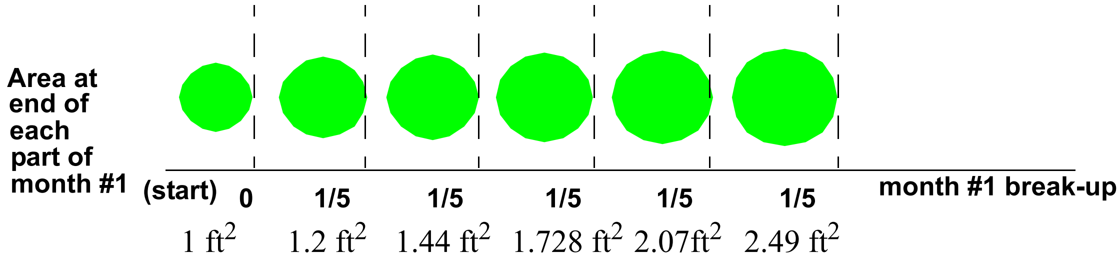

FIGURE 3. Break up first month into 5 equal periods at 100%/5 = 20% per subdivision

Breaking up that first month into a lot of subdivisions

The math of this is to look at the area at the end of that *first* month as being generated by algae growing at a 100% rate at *every instant of time*. To approximate this I can break up that *first* month into finer and finer subdivisions, say: 2, 5, 50, 100, 1000 and see what I get.

Table of areas at end of the *first* month based on the number of subdivisions of that first month.

TABLE 1. Breaking up the first month into many periods

Number of subdivisions of the first month	fractional growth rate	area at end of month #1 , ft^2
2	$(1+ 1.0/2)^{2}$	2.25
5	$(1 + 1.0/5)^5$	2.49
50	$(1 + 1.0/50)^{50}$	2.69
100	$(1 + 1.0/100)^{100}$	2.70
1000	$(1 + 1.0/1000)^{1000}$	2.716
(effectively, a continuous time breakup of that first month)	limit $(1 + 1.0/n)^n$ as n-> Infinity = 2.71828 $= e^{1.0} = e$ Note: the symbol "e" is the shorthand for the limiting expression	2.71828

A Super Important Result - 'e' is *the* natural growth factor

Actually, I have been heading toward this result since I first started the algae example.

> ***Continuous growth is expressed by **powers** of the special constant "e" where e = 2.71828.*

To repeat a definition from the table above:

e^1 = Limit $(1 + 1/ n)^n$, as n -> Infinity

= 2.71828, and arises very naturally as an expression of continuous growth.

As I will explain below, if the growth rate is 'r', where r is some decimal like 0.10,then the growth factor will be:

$\mathbf{e^r}$

for a single period, and if compounded over several periods 'p', then the growth factor is:

e^{r*p}

As mentioned, I will explain all this in the following sections.

Now, Continue that Growth Rate over 5 Months

We have just calculated that the continuous growth for *one* month (the first month) amounted to an area of 2.71828 ft^2, having started with an area of "1". What about the whole five months though? Think of it this way: we broke up the 100%% growth rate into many many subdivisions for that first month, now do the same for all five months, then compound that. The equation shows that first months' growth rate compounded five times. (See also: Figure 4, "Graph of the exponential function for continuous growth at 100% for 5 months," on page 6).

An easy way to calculate the size after 5 months is to recognize that the compound rate for that first month is itself compounded 5 times, as shown next:

$(e^{1.0})^{5}$

$= e^{1.0*5}$

$= e^5$

$= (2.71828)^5 = 148.4\ ft^2$

Compare this with my original calculation where I used only the starting area to calculate the increase for the following months and got 32 ft^2.

Sneak Preview of the Connection between Exponents and Natural Logs

As a preview of things to come, you can see that 148.4 comes from 'e' raised to the 5th power. That exponent, '5', is special because it represents what is called the *natural log* of 148.4. This is true in general, the natural log of *any* number is simply the exponent you need to apply to 'e' to get that number. The rest of the analysis consists of tables or calculators to find that value. For example, if I had the number 7.38906 and asked for its natural log, I could look up in a table the exponent, that when applied to 'e', would give that number. Turns out that I picked 7.38906 to be e^2 so that the natural log is easy to get, it's the exponent '2'. This exponent '2' is conventionally written as:

Ln[7.38906] = 2

The symbol 'Ln' is the conventional way to write 'natural logarithm' but is a bit shorter. In words, this equation says: The natural log of the number 7.38906 is the nunber 2, which is actually the exponent to which 'e' must be raised in order to get 7.38906. So, once again, its helpful to think of a *logarithm as an exponent*, since that's what it is!

A Graph of the Algae Area Under a Continuous Growth Rate of 100%

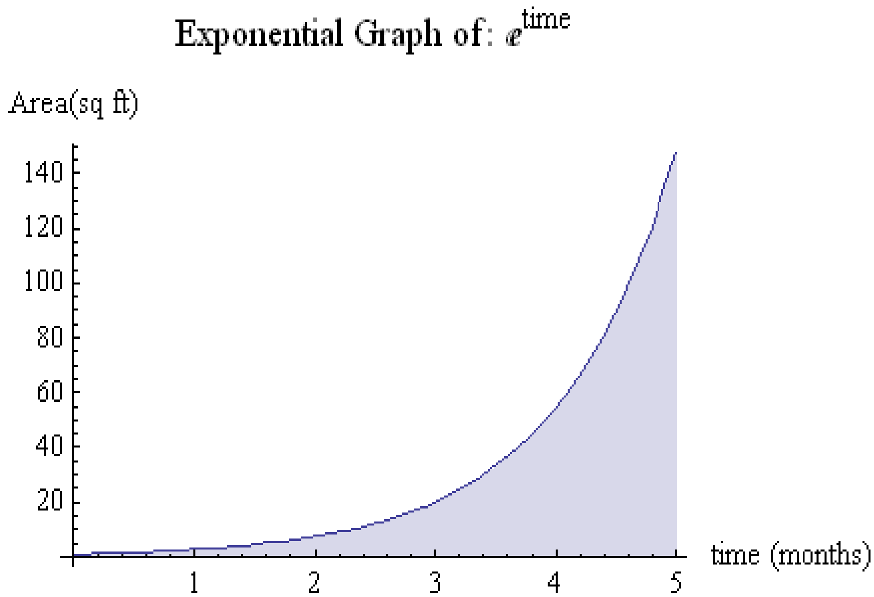

FIGURE 4. Graph of the exponential function for continuous growth at 100% for 5 months

Relating this result to other growth rates (works for money as well!)

If I want to calculate an accurate 5-month final area for my algae using a different growth rate, it's now dead simple:

Calculate the final area at the end of 5 months at a growth rate of 10%:

This is interpreted as continuous growth through the first month at 10%, compounded over 5 months.

A[end of 5 months] = $e^{0.1 * 5} = e^{0.5}$

= 1.65

Calculate the final area at the end of 5 months at a growth rate of 20%:

This is interpreted as continuous growth through the first month at 20%, compounded over 5 months.

A[end of 5 months] = $(e^{0.2})^5 = e^{1.0}$

= 2.71828

Calculate the final area at the end of 5 months at a growth rate of 25%:

A[end of 5 months] = $(e^{0.25})^5 = e^{1.5}$

= 4.48

Calculate the final area at the end of 5 months at a growth rate of 35%:

A[end of 5 months] = $(e^{0.35})^5 = e^{1.75}$

= 5.75

If the Compounding Period is Stated in Years

Often, economics problems will be stated in terms of years rather than months as I did for the algae. In this case the descriptions will say something like - the growth rate is 10% per year compounded annually. In this case we simply substitute years for months and get for, say, a period of 5 years:

One initial unit has grown after five years to: $e^{0.1 * 5} = e^{0.5}$

A Technical Digression (or, how did you get that)?

Given that we know that

$\mathbf{e^1}$ = Limit $(1 + 1/ n)^n$, as n -> Infinity, why is it true that $\mathbf{e^r}$ = Limit $(1 + r/ n)^n$, as n -> Infinity?

If you consider (Limit $(1 + 1/ n)^n)^r$ = Limit $(1 + 1/ n)^{n * r}$

and Limit $(1 + r/ n)^n$, as n -> Infinity, and apply the binomial theorem to both, you will see that both expansions equate as n gets larger and larger. So, if this compounding is carried out over ‘p’ periods, we would get:

e^{r*p}

Net Present Value

We are now able to calculate another economic measure called Net Present Value (NPV). This measure requires us to work backwards from a given future sum to what it must have started out as, given a known growth rate. For example, consider the cash flow line shown below, with compound interest of 10%, compounded yearly. Today’s worth, of a promise of $1210 two years in the future is $1000. That $1000 is the *Present Value* of that Future Promise. That is, today I would trade $1000 for a note that will be worth $1210 two years in the future, if interest stays at 10%. If interest goes up during those two years, I will make more than $1210, if it goes down, I will collect less.

start amount $1000 — 10% growth — $1100 — 10% growth — $1210

0 — 1 — 2

$1210 = 1000 * (1.1)^2$

Future Value = Present Value * growth factor

Future Value/growth factor = Present Value

Assume you could time travel 2 years into the future and you checked your super-savings account. It shows $1210 (this is Future Value) and you also know that interest has been a steady 10% over those two years and your bank compounds annually. What must you have started with two years ago? $1000. Now travel back two years, That $1000 is the present value of a cash flow that *will result* in $1210 two years from now.

Summary

Ok, I hope I have convinced you that continuous compounding depends on the special constant ‘e’ raised to an exponent equalling the rate per period times the number of periods. Once you understand the growth concept you can work backwards and calculate the starting amount, given that you know a future value and the rate it was compounded with.