

Understanding the Binomial Distribution

Introduction

The purpose of this note is to give you insight into the *Binomial Distribution*: context, analysis, and calculations. What I think is difficult when first coming to this distribution is wondering: what's it for, what questions are o.k. to ask, how to ask them, and finally, how to answer them? So, let me say a few words about the kinds of questions the Binomial Distribution is intended to answer and then go through some semi-graphical explanations of where it came from and how to calculate using it.

The next section, Cut to the Chase, is for those who need to calculate binomial probabilities right now, and may choose to read the other sections later.

Cut to the Chase

Notation

Let me use the notation 'p' for the probability of drawing/observing an S (success), and 'q = 1-p', the probability of drawing/observing a failure, F. Let 'x' stand for how many Ss I contemplate/draw/observe out of a sample size of 'n'. So, 'n-x' would be the number of Fs drawn/observed. (That is, if I observed 8 successes out of n=10 draws *then* I would also understand that there were 2 failures). The symbol '!' denotes factorial. So, for example, $8! = 8*7*6*5*4*3*2*1 = 40,320$

Then, the computing formula you need to use is:

Binomial Probability Density Function[n,p,x]: $n!/(x! (n-x)!) p^x q^{(n-x)}$

Example: if p = 0.9, q= 0.1, n=10, x = 8

$$10! / (8! * (10-8)!) * 0.9^8 * 0.1^2 = 0.1937 \sim 0.194$$

This is the probability of observing exactly 8 successes out of a sample size of 10. The probability of observing an individual success is taken to be 0.9 (90%) and is constant from selection to selection for each element comprising the sample.

Example 1: What Can You Expect to Find?

Suppose you *are* Quality Control and are tasked with inspecting 100 complicated, very hard to make, mechanical pencils. Manufacturing assures you that 90% are good (and so 10% are defective/not acceptable). This says, if you believe manufacturing, that the chance of picking a *single* good pencil from a big pile of pencils, is 9 out of 10, (90%), or what is the same, the chance of picking a bad one is 1 out of 10, (10%). O.k, so far so good. Now, a harder question is, what is the chance of finding, say, exactly 8 good ones from a pile of 10, with the other 2 defective? (This turns out to be about 19%). Or, how about wanting to know the chance of finding any other number of good ones in that sample of 10, starting from 0 and going to 10? (Finding those chances for each possible outcome means that you have found the Binomial Distribution, see the table below). Note that I play fast and loose with three equivalent descriptions in this tutorial. When I say the 'probability' of something occurring, that is shorthand for saying, 'the long run relative frequency of occurrence of

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some behavior, assuming it settles down to some fixed number in a time period useful for the analysis at hand'. So, in the discussion above, when manufacturing stated that 90% of the pencils were good, they mean, that in the long run, you can expect to see a relative frequency of 9/10 of good pencils. An even shorter way to say *probability* of an occurrence is simply to say the *chance* of it occurring.

Here is the Binomial Distribution of those chances/probabilities

Suppose you decide, due to time and testing constraints, to inspect only 10 pencils, chosen arbitrarily from the pile of 100. This means that the *sample size* is 10 pencils, with the probability of a good pencil being 90%. Note: to translate this description to standard statistical language, the word 'good' is taken to mean 'success' while a defect is taken to mean 'failure'. Below I have calculated the probability of observing 0 successes, 1 success, 2 successes, . . . 10 successes.

To match standard text books, I have included the somewhat technical notation $P[X=x]$ as well. This is read as the probability of observing the real number 'x' representing the number of outcomes the event, Success, as mapped by the set function X, defined over the event space. There, now don't you feel better!

Note that I haven't told you how to do these calculations yet, that comes later in this tutorial, unless you have read Cut to the Chase!).

Number of Good Pencils (Successes) observed out of 10 pencils	Probability of This Number of Good/Success Pencils (Long Run Relative Frequency of Occurrence of This Number) $P[X = x]$
0	$P[X=0] \sim 0$ this means that finding NO good pencils out of 10 is extremely remote and so I have put down a 0 here. (For you math mavens the actual probability is: $1. \cdot 10^{-10}$)
1	$P[X=1] \sim 0$ (this is the probability of finding exactly 1 good, with probability on the order of $\sim 10^{-9}$)
2	$P[X=2] \sim 0$ (same comment as above $\sim 10^{-7}$)
3	$P[X=3] \sim 0$ (same comment as above $\sim 10^{-6}$)
4	$P[X=4] = 0.00014$ this is the probability of observing exactly 4 good out of 10
5	$P[X=5] = 0.0014$
6	$P[X=6] = 0.011$
7	$P[X=7] = 0.057$ this is the probability of observing exactly 7 good out of 10
8	$P[X=8] = 0.194$
9	$P[x=9] = 0.387$
10	$P[X=10] = 0.349$ this is the probability of observing all 10 good out of 10
Number of Good Pencils Observed	Probability of This Number Being Observed

:Finally, you will want the answers to questions like: what are the chances of finding *at least* 8 good ones, or *less than* 3 defective ones out of 10? Once you have the information in the above table, you can answer these two questions pretty easily just by addition. So, to find the chance of at least 8 good ones, that would be the sum of exactly 8 + exactly 9 + exactly 10 = $(.194+.387+.349) = 0.930$. You could also express this question by saying: "what is the probability of observing 8 or 9 or 10 good pencils from a sample size of 10, given the probability of occurrence of a success as 0.90".

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To find the probability of observing *less than* 3 defectives, I realize that this is asking for the probability of observing 2 or less defectives which, in turn, can be thought of as *at least* 8 good ones, which we just answered. You may want to think of the question this way.

The chance of observing one of the outcomes is unity. That is, for sure you are going to see one of these outcomes: 0, or 1, or 2, or . . . or 10 good pencils. So if I note each of these probabilities as $P[0]$, $P[1]$, . . . $P[10]$, then if I add them up I get unity='1'. That is,

$$P[0] + P[1] + P[2] + \dots + P[9] + P[10] = 1$$

Now you can see how to figure out any question involving the probabilities of the number of successes (or failures).

The word *distribution* in *Binomial Distribution* then is telling you, for each number of good ones you *might* consider, what is the expected frequency (probability) of seeing exactly that many. To actually do these calculations, you will need to understand and use the Binomial Probability Density Function (PDF). I explain this next.

Setting the Context for the Binomial Distribution

The Binomial Distribution is the most important *discrete* distribution around and so I'll take a while to explain its properties. The purpose of using the binomial distribution is to calculate *the probability (long run relative frequency)* of seeing a *certain number of successes (or failures)*, given that you drew a certain sized sample, and that the probabilities of success and failure stay the same from draw to draw. Each draw is assumed independent of any other draw.

The Binomial Distribution context is that you know how to classify a set of entities into a population containing two categories based on some factor: one category is called a success (S) and the other is called a failure (F). [Note: if you can classify some property of a set of entities into two categories, you can let the S and F stand for anything. Don't get hung up on the words 'success' and 'failure', those were a consequence of early usage].

Examples: Inspecting a set of products and categorizing each one as 'Good (S)', 'Bad'(F), like the pencil example above; classifying invoices into those with no errors (S) and those with 1 or more errors(F); perhaps measuring acceptable curing time of concrete test rods, with 48 hrs. equated to (S) and 128 hrs classified as (F); finally, a population of people characterized as having voted in the last election or not. All these examples assume the other conditions hold for a binomial distribution, as discussed initially.

Notation

Let me use the notation 'p' for the probability of drawing/observing an S, and 'q = 1-p', the probability of drawing/observing a failure, F. Let 'x' stand for how many Ss I draw/observe out of a sample size of 'n'. So, 'n-x' would be the number of Fs drawn/observed. (That is, *if* I observed 30 successes out of 100 draws *then* I could conclude that there were also 70 failures).

The Phases of Binomial Distribution Calculations

Here is the formula I would like to take apart and then show how it is put back together:

Binomial probability distribution density function (PDF)

Binomial Probability Density Function[n,p,x]: $n!/(x! (n-x)!) p^x q^{(n-x)}$

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I have labeled the steps for doing the breakdowns and recombination as *phases*. You will find diagrams and numeric examples to illustrate each phase.

O.K, as an introduction to the phases idea, let me calculate the probability of seeing, say, 3 successes and 2 failures (in any order) in a sample size of 5, given a success rate of $p = 2/3$. Note, this is the usual way these questions are asked: *What is the probability of observing a certain number 'x' of successes and '(n-x)' failures, in 'n' trials/draws, given a success frequency of 'p'.* Often you will want to go on and ask for the probabilities of other possible outcomes, for example, what is the probability of getting *less than* a certain number of successes/failure or *more than* a certain number. For now though, I am going to calculate the probability of a particular number of successes out of a specified number of draws, given the probability of drawing an outcome representing an *individual* success.

1. *Phase I:* Calculate the relative frequency of a particular outcome pattern. A pattern consists of a certain number of Ss and Fs, *in a particular order*. (For example, what is the long run relative frequency (probability) of observing, say, {S F S S F}, *in that order*, in a sample of five draws, given p and q). The answer in general would be: $p^*q^*p^*p^*q$ which in this case is: $2/3 * 1/3 * 2/3 * 2/3 * 1/3 = .033 = 8/243$
2. *Phase II:* Calculate the Binomial Coefficient of each outcome pattern found in Phase I. This is equivalent to asking: for each of those ordered patterns discovered above, how many *un-ordered patterns* are there, consisting of the same number of Ss and Fs. Using the example in Phase I, how many outcomes have 3 Ss and 2 Fs, in any order? The answer in general is: $n!/(x! * (n-x)!)$ where n! means n-factorial, x! means x-factorial, and (n-x)! means n-x factorial. Specifically for the outcome above, it is: $5!/(3!*2!) = 10$. That is the binomial coefficient for this pattern. This just tells you how many patterns have 3 Ss and 2 Fs.
3. *Phase III:* Multiply the Binomial Coefficient found in phase II by the probability of a particular outcome pattern found in phase I. For example, the probability of observing 3 S and 2 F in 5 draws, in any order, is: $10 * p^*q^*p^*p^*q = 10 * 2/3 * 1/3 * 2/3 * 2/3 * 1/3 = .33$.

Ok, the combined phases yields the binomial distribution probability density function

Binomial Probability Density Function $[n,p,x]: n!/(x! (n-x)!) p^x q^{(n-x)}$

Using this formula answers the question: *What is the probability of observing 3 successes out of 5 draws, given the probability of success is 2/3.* (assuming all the other constraints of the binomial distribution are satisfied). In essence, we found the probability of a particular pattern consisting of 3 S and 2 F and then multiplied that probability by the total number of patterns having 3 S and 2 F, in any order.

An Easy Example, Urns and Hikes

Let me step back and begin with a very easy example to show how this distribution works and how the phases can be used to understand it.

Note: I will often substitute 'relative frequency' for the word probability since a probability *is* a long run relative frequency, so watch for that substitution!

An Analogy for the Academically Minded Folks

O.k, here goes: assume I have two grecian urns with three balls in each. The first urn has 2

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balls marked S_1 , S_2 , and 1 ball marked F_1 . The second urn has 2 balls marked S_3 , S_4 and one ball marked F_2 . (This means that the relative frequency (probability) of drawing an S, is $2/3$ while the relative frequency of drawing an F is $1/3$). Now I walk up to urn 1, reach in and pull out a ball (without looking). I could have gotten any one of the three balls. Now, holding the first ball in my left hand, I walk over to the second urn and pick a ball with my right hand. After this second draw I could be holding, in both hands, one of 9 patterns. (see Figure 1, "Simple Example: Phase I diagrams for $n=2$, and $p=2/3$," on page 6.) What I have here is a sample size of '2', two independent draws, with the same probabilities at each draw.

One of several binomial questions to ask here is: Given that my sample size is 2, (I have drawn twice), *What is the probability that I have drawn/chosen one S (which means that I have chosen 1 F as well)?* We will see that this answer is: $2 * 2/3 * 1/3 = 4/9$. Other usual binomial questions to ask are about the probability of observing either 0 or 1 or 2 Ss in the sample of size 2 (that is, two draws). See the diagram below for a picture of this outcome.

Another Analogy for you Hikers Out There

Another vivid analogy here is to think of tracing out branching hiking 'paths'. Suppose you are in the wilderness, at a trail-head (that is, at the start of trails) and you see that there are two kinds of paths: A Fun (F_1) path, which is easy with Dairy Queens along the way, and two marked Strenuous (S), S_1 and S_2 paths which involve steep cliffs, hungry bears, and raging rivers. You take one. After $1/2$ mile, each path branches again into three possibilities, an F_2 and 2 S paths (S_3 and S_4). Visualize this in your mind and you will see you could have taken 9 total paths, with 9 different end points. Those are the 9 patterns I am talking about.

One of several binomial questions to ask here is: *What is the probability that I have/will have followed/chosen two S paths (which means that I have chosen 0 F paths)?* We will see that this answer is: $1 * 2/3 * 2/3 = 4/9$. (We will talk later about what would motivate you to take two S paths!) So, the usual binomial questions to ask are about the probability of observing either 0 or 1 or 2 Ss in your path choices.

Phase I Questions and Calculations for the Urn/Hiking Example

What is the relative frequency of observing each of the *ordered* outcomes listed below, where there are two draws/choices and the relative frequency of success (S) is $2/3$ and relative frequency of failure(F) is $1/3$?

How many times will I see (check out the diagram):

2 S, in that order? (from the diagram below, this is $2/3 * 2/3 = 4$ out of 9)

1 F and 1 S, in that order? (from the diagram below, this is $1/3 * 2/3 = 2$ out of 9)

1 S and 1 F, in that order? (from the diagram below, this is $2/3 * 1/3 = 2$ out of 9)

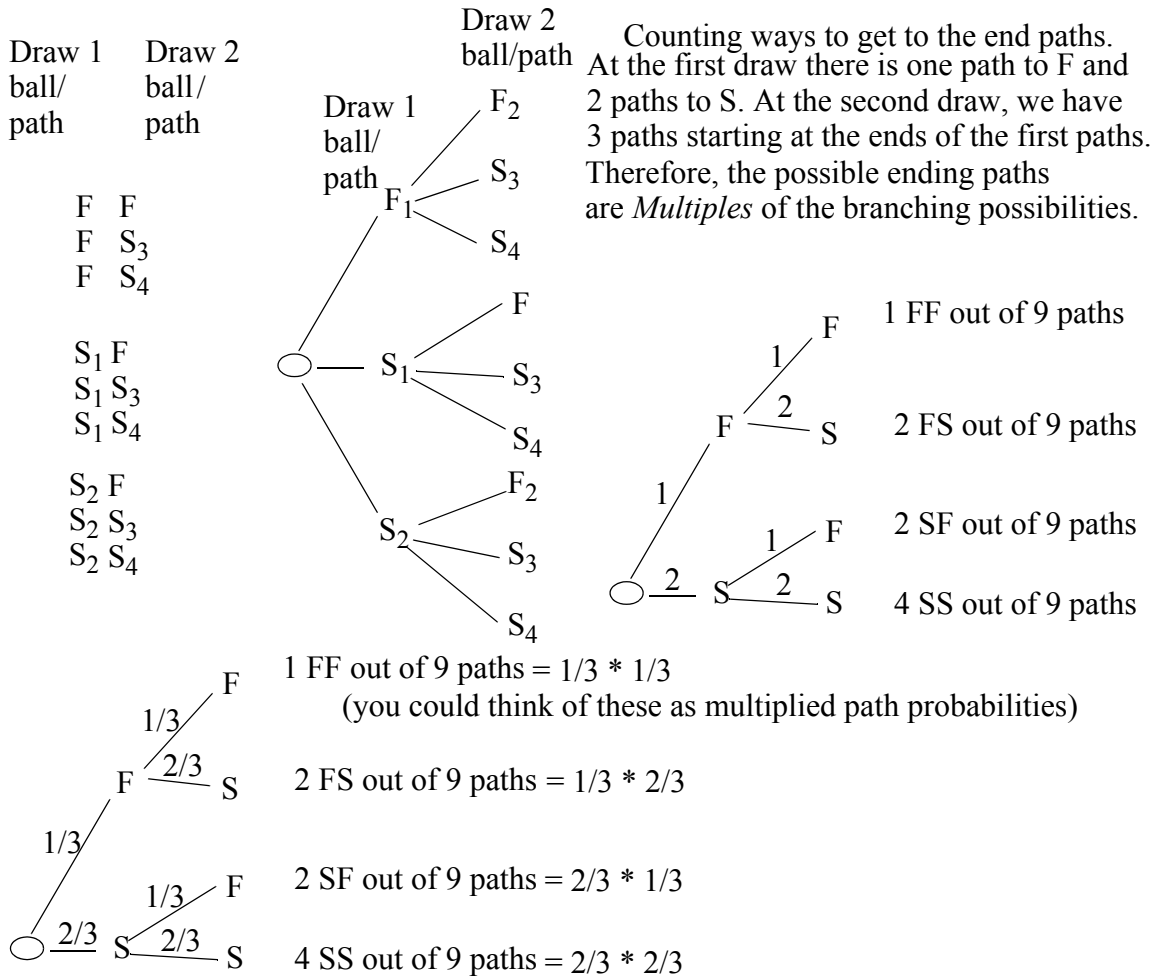
2 F, in that order? (from the diagram below, this is $1/3 * 1/3 = 1$ out of 9)

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Answering **Phase I Questions**: What are the relative frequencies of a particular *ordered* outcome?

(Note: Phase II will answer the question of how many *un-ordered* patterns)

After 2 draws I am holding one of these 9 outcomes.



relative frequency of obtaining:
 0 S, is $1/9$
 1 F and 1 S, in that order = $2/9$
 2 S, in that order = $4/9$

Now we know how to get the relative frequency of a particular ordered outcome. The next task is to calculate how many such outcomes there are, not taking into account order, that's Phase II.

FIGURE 1. Simple Example: Phase I diagrams for n=2, and p= 2/3

Phase II Questions and Calculations for the Urn/Hike Example

Given that I know the chance of seeing any particular ordered outcome, that is, with a certain number of Ss and Fs, in a particular order, how can I find how many outcomes have this same number of Ss and Fs?

From the diagram you can already see, in our simple example, that there are 2 expressions of the pattern of 1 F and 1 S, if you don't keep them in order. Also you can see there is only 1 expression of the FF and SS patterns. These coefficients, {1, 2, 1}, are the Binomial Coefficients. So, for n=2, p = 2/3 we could use the binomial coefficients formula to get:

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Binomial Coefficient[2,1]= $2!/(1! * (2-1)!) = 2$ (we will look at how this is done below in the Phase II procedure).

Phase I Questions and Calculations in General

Remember, this phase is in response to a request to find the probability of observing a certain number of successes, 'x', in a sample of size 'n', given a probability of success of 'p' and a specified order of occurrence.

To calculate the relative frequency of a particular ordered sequence of x Ss and (n-x) Fs, with relative frequency of S = p.(Keeping to our example: n = 5, x=3, p=2/3 as an illustration)

.work as follows:

temp1 = multiply p by itself x times: $p * p * p$

temp2 = multiply (1-p) by itself (n-x) times: $(1-p) * (1-p)$

So, probability of observing x Ss and (n-x) Fs in a certain order is:

$$\text{temp1} * \text{temp2} = p * p * p * (1-p) * (1-p) = p^3 (1-p)^2$$

Using the example we would get:

$$\text{temp1} = 2/3 * 2/3 * 2/3$$

$$\text{temp2} = 1/3 * 1/3$$

probability of observing 3 S and 2 F in a certain order would be:

$$2/3 * 2/3 * 2/3 * 1/3 * 1/3 = 8/243$$

Phase II Questions and Calculations in General

To find the number of outcomes showing 'x' successes in 'n' draws, we calculate the binomial coefficient as:

*****Binomial Coefficient Formula*****

Binomial Coefficient[n,x]: $n!/(x! * (n-x)!)$

For example, if n = 5, x=3, then the question of how many patterns would show 3 S, and 2 F (in any order), could be calculated as:

Binomial Coefficient[5,3]: $5! / (3! * (5-3)!) = 5 * 4 * 3 * 2 * 1 / (2 * 1 * 3 * 2 * 1) = 10$ such patterns

For 4 S and 1 F, I would calculate:

Binomial Coefficient[5,4]: $5! / (4! * (5-4)!) = 5$ such patterns

How do I Understand these Binomial Coefficients?

Let me stick with the n= 5 example, with 3 S and 2 Fs, and show you one way to understand the associated binomial coefficient. Of course, we could use the Binomial Coefficient Equation and immediately calculate: $\text{Binomial}[5,3] = 5!/(3!*2!) = 10$, as was done above. True enough, but where does this '10' come from?

Let me start by putting subscripts on the three successes S, as well as the two Fs. Let me start with one particular pattern for S and Fs, say, {S1 S2 F1 F2 S3}. The question is, how many rearrangements are there having 3 S and 2 Fs? We obviously have one, how many

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more are there? I will start by calculating the total number of possible arrangements of the three distinct S elements and then factor out duplicates (it's as if, after all the possible arrangements were lined up, then the subscripts got rubbed off somehow). With the subscripts rubbed off, there are an awful lot of indistinguishable patterns, yes? How many?

Take S_1, S_2, S_3 how many ways can I rearrange them?

Well, I can put one of those three first, two are left to put second and the last is determined for a total of $3 * 2 * 1 = 3! = 6$. These are called the *permutations* of the three elements. Now think of the subscripts as rubbed off. What you see are three Ss, but there are really 6 distinct patterns in there

$S_1 S_2 S_3$

$S_1 S_3 S_2$

$S_2 S_1 S_3$

$S_2 S_3 S_1$

$S_3 S_1 S_2$

$S_3 S_2 S_1$

Now consider the two Fs, F_1 and F_2 . How many ways can they be rearranged? Looks like 2 ways = $2 * 1 = 2! = 2$

$F_1 F_2$

$F_2 F_1$

Now for the total number of arrangements of 5 distinct elements you can figure that it would be: $5*4*3*2*1 = 5! = 120$ possible arrangements. That is, one of the 5 elements could come first, then one of the remaining 4 could come next, one of the remaining 3 could be in the next position and so on.

So, among the 120 distinct patterns, if all the subscripts were considered, there are $3! * 2!$ duplicates. So, the number of patterns with 3 S and 2 Fs, (with the subscripts rubbed off)

$$120 / (3! * 2!) = 5! / (3! * 2!)$$

If you plug in $n = 5$, $x = 3$ and $n-x = 2$ in the binomial coefficients formula, you get the same number.

Phase III, Multiplying Phase I and Phase II Values

This phase is real easy, just multiple phase I and phase I values together and that's it, as shown next.

Probability of 3 Successes (2 Failures): Combining Phase I and Phase II

Let me combine phase I and phase I to calculate the actual probability of observing 3 successes in a sample of 5 draws. Assume $p = 2/3$ and $q=1/3$.

Phase I: What is the relative frequency of seeing 3 S and 2 F", in a particular order. So the answer to the phase 1 question would be $2/3 * 2/3 * 2/3 * 1/3 * 1/3 = 8/243$

Note that every pattern of 3S and 2F would have this same value, 8/243.

Phase II: How many patterns would show these 3S, and 2 F (in any order). This is calculated as:

$$5! / (3! * 2!) = 5*4*3*2*1 / (3* 2* 1* 2*1) = 10 \text{ such patterns}$$

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The *probability* of seeing 3 S and 2 F in a sample of size 5, then is (combining phase I and phase II):

$$10 * 8/243 = 0.33$$

This says that I have a 33% chance of seeing 3 S in a draw of 5 units. (or equally, a chance of 33% of seeing 2 Fs in a draw of 5)

Finally, the Binomial Distribution for $n=5$, $p=2/3$

Formally, if I called X the random variable associated with the number of Ss observed in a draw of 5 units, with p chance of success at each draw, I would write:

$$P[X=0] = 1 * (1/3)^5, \text{ that is, all five outcomes are Fs} = 0.004$$

$$P[X=1] = 5 * 2/3 * 1/3 * 1/3 * 1/3 * 1/3 = .041$$

$$P[X=2] = 10 * 2/3 * 2/3 * 1/3 * 1/3 * 1/3 = .164$$

$$P[X=3] = 10 * 2/3 * 2/3 * 2/3 * 1/3 * 1/3 = .329$$

$$P[X=4] = 5 * 2/3 * 2/3 * 2/3 * 2/3 * 1/3 = .329$$

$$P[X=5] = 1 * 2/3 * 2/3 * 2/3 * 2/3 * 2/3 = .131, \text{ all five outcomes are Ss}$$

Showing the Binomial Distribution for $n=5$, $p=2/3$

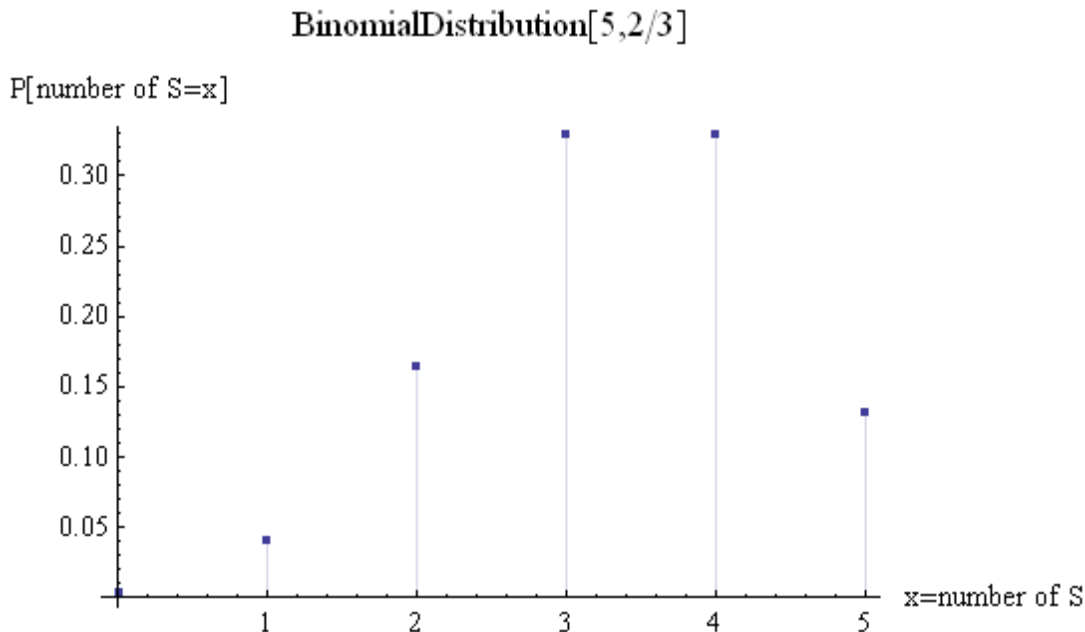


FIGURE 2. Plot of Binomial Distribution for $n=5$, $p=2/3$

Parameters of the Binomial Distribution

The Mean of the Binomial Distribution = $n * p$

The importance of the simple example above shows up when you are dealing with situations where you know the relative frequency of occurrence of a particular kind of element in the distribution and you wish to predict how often you will find one or more of this kind

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in a sample. In the examples up to now, the relative frequency of drawing an 'S' ball is 2/3. So a common kind of question would be: if I randomly draw 30 balls from a very large collection of balls that have a ratio of 2 Ss to 1 F, how many S balls will I see?

It may be pretty clear in this first question that you could just take the average as your best guess and get: $30 * 2/3 = 20$ since 2/3 of the balls are 'Ss'. Your intuition is right, that is the mean of the binomial distribution. (you might want to refer to the tutorial *Normal Rules* for a more detailed explanation of Means and Variances)

Deriving the Mean of the Binomial Distribution (Optional Material)

Recall that we initially introduced the density function as:

Binomial Probability Density Function[n,p,x]: $n!/(x! (n-x)!) p^x q^{(n-x)}$

Abbreviating this to $p(x)$ I can write:

$$p(x) = n!/(x! * (n-x)!) p^x q^{(n-x)}$$

The mean of the binomial distribution is then defined by *eq1* below to be:

eq1: $\sum x * p(x)$ where the summation goes from $x=0$ to $x = n$

Step 1: factor out $n!p$ from *eq1*., since they are constants, and you are left with

eq2: $n!p \sum (n-1)! / ((x-1)!(n-x)!) p^{(x-1)} q^{(n-x)}$ where

the summation now goes from $x=1$, $x=n$

substitute $m=n-1$ and $z = x-1$ into *eq2*, and you will get:

eq3: $n!p \sum m! / (z!(m-z)!) * p^z q^{(m-z)}$ where the sum goes from $z=0$ to $z=m$ but that sum is a binomial sum itself and equates to "1"

this leaves the mean of the binomial distribution as **$n * p$**

Variance of the Binomial Distribution: $n * p * q$

The variance is

$$n * p * q$$

The algebra is a bit complicated but you can derive it by noting that:

$$\text{Variance} = -(n * p)^2 + \sum x^2 * p(x)$$

This in turn comes from the definition of variance as

$$\text{ExpectedValue}(X - n * p)^2$$

where X is the random variable equalling the number of outcomes, from zero to 'n'.

Example of a Larger Sample Size, Just for Fun

Consider the Binomial Distribution for a Sample Size of 30 with Success Probability of 2/3. The diagram below shows a picture of the probability of observing a certain number of successes, given a sample size of 30 and a probability of drawing an 'S' at 2/3.

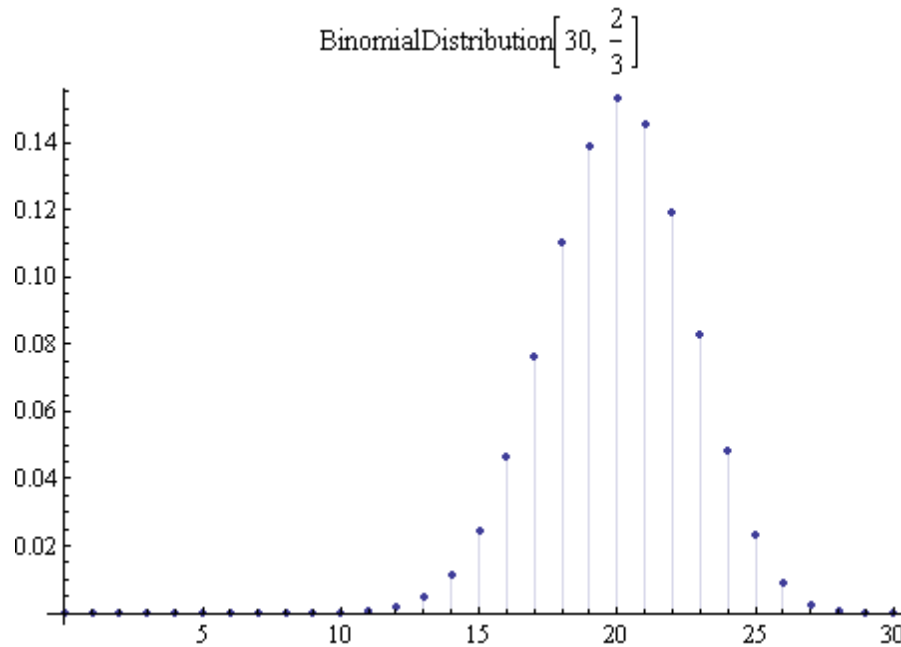


FIGURE 3. Binomial Distribution of sample size 30 with probability of success at 2/3

Summary

We have looked at how to break up the equation for the binomial probability density function into two components, path probabilities and the number of those paths. The path probabilities will follow from multiplying out the number of success probabilities by the number of failure probabilities. The number of paths will be given by the binomial coefficients formula. Examples were given of these analyses and calculations.

As a final note, this last picture, “Binomial Distribution of sample size 30 with probability of success at 2/3” on page 11, is starting to look a lot like a normal curve, no? Your suspicions are correct. At large sample sizes, and for probabilities not too close to zero or one, we will be able to use the Normal Curve Approximation.

Hope all of this helps you to learn.

Cheers

Rob Rucker [2008-02]

References

Rucker, R.,(2007) *Normal Rules rev 2*, available from the author.

Rucker, R.,(2008) *Business Trend Analysis*, available from the author.