

Physics: Conservation of Momentum and Mechanical Energy [2009-04-27]

■ Introduction

This tutorial illustrates two basic principles, used in sequence, to solve a common physics question with practical applications. The question is: how to find the speed of an incoming bullet? Answering this question will illustrate the interplay of conservation of momentum and conservation of mechanical energy. I have used the ideas of the fine text, *Physics* by Wolfson and Pasachoff. Of course, the ideas are ultimately attributable to Newton and Galileo.

Note: This whole document is written using the math package *Mathematica*. There is an accompanying 'live' notebook that I will send to anyone interested. (It requires *Mathematica* or the *Mathematica* Player to manipulate it though). This PDF file you are reading is the static translation of that *Mathematica* notebook.

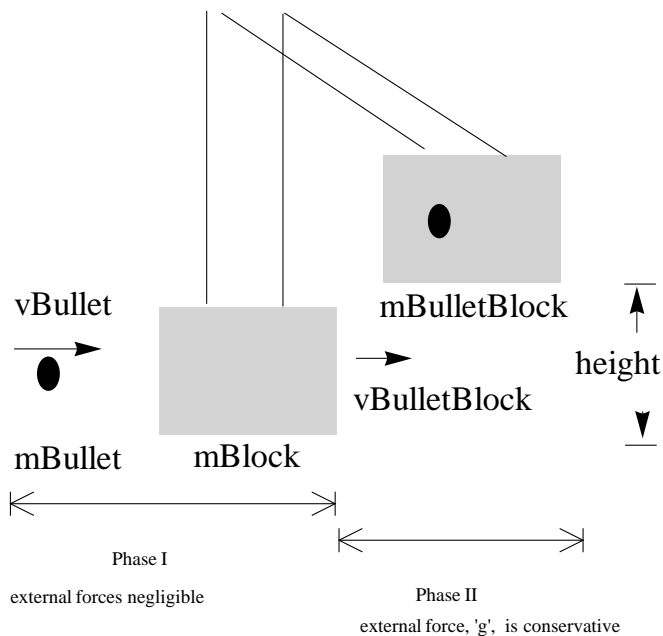
As an aside, this tutorial also shows a few features of the *Mathematica* package that can be used to solve math/physics problems.

Finally, you will notice that I state these principles several times, in slightly different ways. I hope that one of these repetitions will resonate with you, my reader.

■ A Speeding Bullet Embeds in a Block - What Was the Incoming Speed?

The set-up is simple, I want to calculate the incoming speed of a bullet by means of measuring the height to which a large suspended wooden block will rise after the bullet embeds in the block.

This set-up is a common approach used to measure the speed of objects. What is nice is that this example illustrates two very important conservation principles: conservation of (linear) momentum and conservation of mechanical energy. Below is a before and after picture of this situation: incoming bullet embeds in large block which swings up to a certain height. The masses involved and the height will determine the incoming speed, but it will take the application of two conservation principles to find that speed.



A Few Background Definitions and Ideas

$F_{\text{netExternal}}$ is the vector sum of all the external forces acting on a designated system, (in this tutorial the *system* is composed of the bullet and block). These definitions are written out below as equations.

P is the sum of all the *momenta* of the designated system, KE refers to the kinetic energy of the designated system while PE refers to its potential energy. Mechanical energy (ME) is defined to be the sum of KE + PE. Note that a potential energy representation *must* be associated with a conservative force, which in this case is the force of *gravity*, symbolized by 'g'. The notation m_i and v_i refer to the mass and vector velocity of component particles. In this tutorial there are only two particles, the bullet and the block and they comprise the *system*.

■ Cut to the Chase

If you don't have time to read the detailed description of this experiment, just read through this next section while referring to the diagram above.

■ Phase I Conservation of Linear Momentum

In the case of negligible external forces acting during this inelastic collision we have during *Phase I* :

$$P_{\text{initial}} = m_{\text{Bullet}} * v_{\text{Bullet}} + m_{\text{Block}} * 0 \quad (* \text{ assuming the block is initially at rest } *)$$

$$P_{\text{final}} = m_{\text{BulletBlock}} * v_{\text{BulletBlock}}$$

Conservation of momentum will require that

$$P_{\text{initial}} = P_{\text{final}}$$

in Phase I, we end up looking for the velocity with which the combined bullet - block mass moves off. That velocity depends on the initial bullet velocity, which we don't know and want to find out. So, we need more information, which is given in Phase II where we invoke another supremely important principle, the conservation of Mechanical Energy.

■ Phase II Conservation of Mechanical Energy

The instant the Phase I inelastic collision is over, and the bullet - block mass moves off with that initial velocity, the external forces are no longer negligible and must be accounted for. The only external force is *gravity*, which is a *conservative* force. (The strings act at right angles to the motion of the bullet-block system and so don't contribute to the Mechanical Energy calculations). Therefore we can write:

$$ME_{\text{initial}} = KE_{\text{initial}} + PE_{\text{initial}} = 1/2 m_{\text{BulletBlock}} * v_{\text{BulletBlock}}^2 + m_{\text{BulletBlock}} * g * 0$$

(this is at the instant that Phase II starts. I have taken the potential energy reference point to be at height ==0 where I declare potential energy to be zero). Note that only *differences* in potential energy have meaning.

$$ME_{\text{final}} = KE_{\text{final}} + PE_{\text{final}} = 1/2 m_{\text{BulletBlock}} * 0^2 + m_{\text{BulletBlock}} * g * \text{height}$$

(this is at the instant that Phase II ends and the bullet-block is at its max height)

■ Summary of Cut to the Chase

From phase I we get the initial velocity, $v_{\text{BulletBlock}}$, with which the bullet - block mass moves off. That implicitly involves the bullet velocity, which we still don't know. However, using principle #2 in phase II lets me calculate what the bullet velocity *must have been* in order to raise the block to that given height.

More General Notation

(The notation below is overkill in this case but, shows how to set up the equations in general).

$$P = \sum_{i=1}^n m_i * v_i$$

$$KE = 1 / 2 \sum_{i=1}^n m_i * v_i^2$$

$$PE = \sum_{i=1}^n m_i * g * h_i , \text{ where } h_i \text{ is the height above a zero reference level}$$

■ Detailed Discussion of the Exercise

■ The Two Principles

$$\text{Principle \#1 : } F_{\text{netExternal}} = \frac{d}{dt} P$$

$$\text{Principle \#2 : Mechanical Energy} = KE_{\text{initial}} + PE_{\text{initial}} = KE_{\text{final}} + PE_{\text{final}}$$

So, principle #1 says that when the net external force on a system is zero ($F_{\text{netExternal}} = 0$), then the total linear momentum must remain constant. That is, when $F_{\text{netExternal}} = 0$, the *time rate of change* of P must be zero (P doesn't change), therefore, P must be constant before and after an interaction. In our case the 'initial and final' pictures during Phase I must show the same total momentum regardless of how the momenta are split up between the participating components. In this particular example, the bullet carries all of the initial momentum while the bullet-block mass carries all the final momentum. They are equal. Further, in an inelastic collision, the external forces can be ignored, being much less than the collision forces. That is why we can say that we will ignore any external forces during the time of collision.

Principle #2 states that if a system is subject to only external *conservative* forces, then its mechanical energy stays constant. That is the case for Phase II.

The force of gravity is the poster child of a conservative force. For example, if I lift that wooden block over my head, the energy used to lift it is now in the form of (gravitational potential energy) PE, and can be recovered by dropping it, thus converting the PE to KE! As another common example, if you stretch an (ideal) spring just a little bit, the Potential Energy (called elastic potential energy) stored in the stretched coils can be recovered as motion (KE) when you let go. So, one says: Mechanical Energy is conserved if only conservative forces are involved. (Yes, it's circular, but that's what it means!)

Ok, back to the illustration: First, and most importantly, let me define the *system* I want to analyze - in this case I want to consider the bullet plus block as my *system*. That way, force interactions between the bullet and block are *internal forces*,

and by *Newton's 3rd law*, will cancel out, leaving only the net external force to determine behavior. Since the net external force on my system is negligible as far as the conservation of momentum goes, I can apply the constant momentum principle and say that the initial momentum, which is due solely to the incoming bullet, is equal to the final momentum of the combined bullet and block.

****Be careful here:** this first conservation law applies because the net external forces are negligible just before and just after the instant of impact. This means that the momentum conservation law determines the speed at which the bullet+block system moves off, at the *instant of impact*. After that initial instant, the *net external force* is no longer zero (gravity kicks in) so the momentum conservation law no longer applies *but*, the external force now is of a special kind, a conservative one, and thus the second principle applies. What we are left with, after the instant of impact, is the applicability of principle #2, a conservative force acting on the bullet-block system.

*The careful reader may also notice that there is another external force on the block + bullet system, namely the *tension force* of the suspending strings, pulling upward. However, this tension force is *perpendicular* to the resultant motion of bullet-block and so doesn't enter into the conservation of mechanical energy equation. This is because the mechanical energy conservation law takes into account only *forces in the direction of motion*, perpendicular forces don't count.

■ Principle #1 Calculating with the Conservation of Linear Momentum

When the bullet strikes the block, all kinds of macro and micro processes are in play, but these, by definition, are internal to the system (bullet + block) as defined. Newton's third law says these internal forces cancel out, in pairs. Further, since there are negligible external forces on the system, linear momentum is conserved. That principle will allow me to determine the speed with which the block and embedded bullet will initially move off. Conservation requires that the initial momentum must equal the final momentum. The final answer here is that the `vBulletBlock` speed is the momentum of the bullet scaled by the sum of the block and bullet masses. What I don't know is the speed of the bullet (`vBullet`), so I need another equation. (See principle #2). But so far, conservation of momentum yields:

Using the Mathematica Package to Do Some Calculations

Note that Mathematica uses a space in place of the '' symbol to indicate multiplication.

■ Applying Principle #1 in Phase I, Conservation of Momentum

This phase of the calculations doesn't get us to the bullet's speed, but to an intermediate result involving both bullet and block. I will then use this result and the second principle to finally get the bullet's speed.

```
Clear[mBullet, mBlock, vBullet, vBulletBlock] (* null out variables *)
```

■ Here is the conservation of momentum equation, equating initial with final momentum

```
eq1 = mBullet * vBullet == mBulletBlock * vBulletBlock;

eq2 = eq1 /. mBulletBlock -> (mBullet + mBlock)
      (* use knowledge that bullet and block become one single mass*)

mBullet vBullet == (mBlock + mBullet) vBulletBlock
```

■ Ok, here is the speed of the combined bullet - block as they initially move off

I use the *Mathematica's* built in **Solve** function to get `vBulletBlock`

```
vBulletBlock = (*solve for the speed which the bullet+block moves off*)
vBulletBlock /. Solve[eq2, vBulletBlock][[1]]

mBullet vBullet
-----
mBlock + mBullet
```

■ Apply Principle #2 in Phase II, Conservation of Mechanical Energy

Once the bullet embeds in the block, the bullet-block mass moves off at speed $v_{\text{BulletBlock}}$ as calculated above. After this moment, conservation of momentum doesn't apply anymore since there is now an *external force* on the bullet-block system, namely, gravity. Since this is a *non-zero net external force* on the system, momentum changes and is no longer constant. Now however, the second principle kicks in: Gravity is a conservative force on the bullet-block system and will conserve any initial mechanical energy (KE + PE). That initial mechanical energy is totally in the form of Kinetic Energy and is due to the speed of the combined bullet-block mass. That energy will be conserved as the bullet-block mass swings up to its maximum height, where the Kinetic Energy is now all converted to (gravitational) Potential Energy.

If I take the zero of potential energy at the initial height of the bullet+block system, and the final height to represent potential energy in the form of $\text{mass} * g * \text{height}$, I can equate the initial kinetic energy (K.E.) to this final potential energy (P.E.) as below. Where 'g' is the acceleration due to gravity in SI units = 9.8 m/s^2

$$g = 9.8;$$

■ Mechanical Conservation Equation: initial KE + PE = Final KE + PE

Note that the initial PE is set to zero since I *chose* the initial height to have zero PE. Therefore the initial mechanical energy is all KE. The final KE is zero since that is the status of the bullet-block mass just as it reaches its maximum height. Therefore the final mechanical energy is all PE. Conservation requires that initial mechanical energy equals final mechanical energy (under the action of only conservative forces)

$$\begin{aligned} \text{eq3} &= \frac{1}{2} (m_{\text{Bullet}} + m_{\text{Block}}) v_{\text{BulletBlock}}^2 = \\ & (m_{\text{Bullet}} + m_{\text{Block}}) * g * \text{height} \\ \frac{m_{\text{Bullet}}^2 v_{\text{Bullet}}^2}{2 (m_{\text{Block}} + m_{\text{Bullet}})} &= 9.8 \text{ height} (m_{\text{Block}} + m_{\text{Bullet}}) \end{aligned}$$

■ First Solution for the vBullet Speed, not too Intuitive!

You will notice that I am using *Mathematica* to carry out these calculations. The result below is what the package delivers. While correct, it is not the simplest way to look at the speed of the bullet.

```
vBullet = vBullet /. Solve [ eq3, vBullet][[2]]
```

$$\frac{1}{m_{\text{Bullet}}} 1.41421 \sqrt{\text{height}} \sqrt{m_{\text{Block}} + m_{\text{Bullet}}} \sqrt{9.8 m_{\text{Block}} + 9.8 m_{\text{Bullet}}}$$

■ Transforming to a more intuitive expression

While the above solution is correct, I want to transform it so that I can plot the effects of different input values of m_{Bullet} , m_{Block} , and height . What I am after is the dependence of the initial bullet speed (v_{Bullet}) on these parameters. That is the point of the manipulations below.

$$v_{\text{Bullet}} = v_{\text{Bullet}} /. \text{Sqrt}[a_x_ + a_y_] \rightarrow \text{Sqrt}[a] \text{Sqrt}[x + y]$$

$$\frac{4.42719 \sqrt{\text{height}} (m_{\text{Block}} + m_{\text{Bullet}})}{m_{\text{Bullet}}}$$

■ Here's what I was after, v_{Bullet} simply expressed

$$v_{\text{Bullet}} = v_{\text{Bullet}} /. ((m_{\text{Block}} + m_{\text{Bullet}}) / m_{\text{Bullet}}) \rightarrow 1 + m_{\text{Block}} / m_{\text{Bullet}}$$

$$4.42719 \sqrt{\text{height}} \left(1 + \frac{m_{\text{Block}}}{m_{\text{Bullet}}} \right)$$

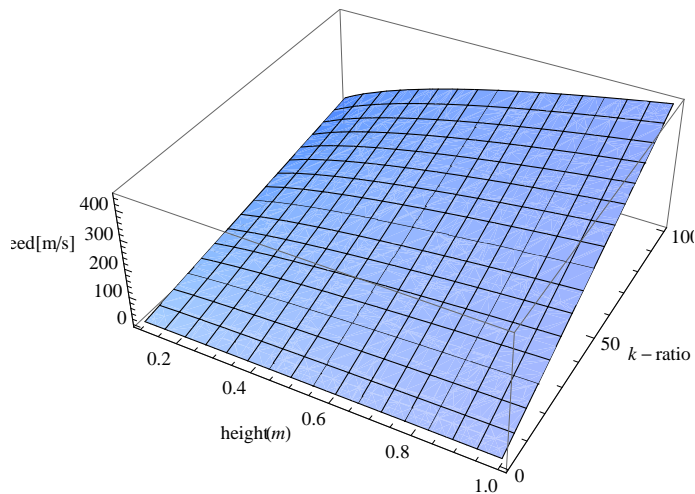
Notice that I now have the v_{Bullet} in terms of height and a ratio of m_{Block} to m_{Bullet} . Finally, let me replace the $m_{\text{Block}}/m_{\text{Bullet}}$ ratio with a parameter 'k'.

$$v_{\text{BulletK}} = v_{\text{Bullet}} /. m_{\text{Block}} / m_{\text{Bullet}} \rightarrow k$$

$$4.42719 \sqrt{\text{height}} (1 + k)$$

■ Profile of v_{Bullet} speed as a function of height, and the ratio of m_{Block} to m_{Bullet}

```
Plot3D[vBulletK, {height, 0.1, 1}, {k, 1, 100},
  AxesLabel -> {height[m], k - ratio, "speed[m/s]"}]
```



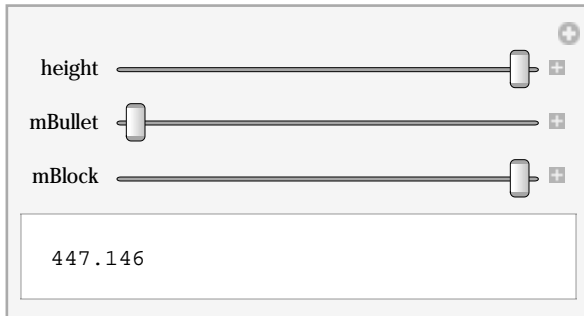
The upper right hand corner is what you would see, if the block swings up 1 meter and the block is 100 times as massive as the bullet. From the diagram you can see the bullet must have been traveling at about 450 m/s, which is about 900 mph.

■ An Alternative Description, using Manipulate

If you would like to manipulate these parameters in a discrete manner and see their effect, you can use **Manipulate**, a built in function.

I think of Manipulate as working around a *set point* for some expression and then using variable parameters to explore around that. For example, the static picture below shows height is set to '1 meter', and m_{Bullet} at 0.1kg, and m_{Block} at 10kg. This combination of parameters matches the same result as in the graph above with $k' = 10/0.1 = 100$, that is, the upper corner of the v_{Bullet} surface.

```
Manipulate[Plot3D[vBulletK, , {height, 0.1, 1}, {mBullet, 0.1, 0.5}, {mBlock, 1, 10}]
```



Summary

In this tutorial I have illustrated two of the most important principles in physics, conservation of momentum and conservation of mechanical energy.

In general, the overall plan in physics is to look for underlying constraints (called invariants) that make analyses and predictions easier (or possible at all!). The preeminent invariant in physics is the conservation of momentum which applies equally to the behavior of atoms as well as galaxies. The next most important principle is the conservation of energy, more generally known as the conservation of mass - energy. (The underlying deepest principle in physics is the conservation of symmetry, but that can wait for another time!). Those two principle are what I was aiming to show in this document. I hope this discussion resonates with you.

Best wishes in your own studies of our world and its behavior - cheers, rob rucker