Vector Operations & Vector Spaces (*Draft 09-26)

This tutorial continues on from the *Trigonometry Basics* tutorial, available on this site. If you are a bit hazy about Pythagoras' theorem, or the Dot Product, take a few minutes and refresh your knowledge with that earlier tutorial. The "Cut to the Chase" section next will get you started quickly though, and has abbreviated discussions of these background techniques.

Cut to the Chase (for those with no time!)

This "Cut to the Chase" section of the tutorial is mercifully brief. I intend this to be background to later tutorials needing vector operations and vector spaces to set up and analyze statistical/quantitative experiments. The follow-on tutorial that relies on this one is *GeoStatistics Part I* on the *milagrosoft.com* site. The section after this one, "*** End Cut to the Chase ***" on page 4, goes much deeper into the same material.

A Vector == list of values (algebraic view), directed line segment (geometric view)

In the diagram below, **A** = {2, 5} is a list of two numbers, 2 and 5, in that order. That's the algebraic perspective. I can also 'place' that list on a 2-dimensional grid and interpret it as a "vector" line segment with **A** starting at {0,0} and going out to {2,5}. That's the *geometric view.* More colorfully, the *tail* of **A** is at {0,0} and its *head* is at {2,5}. For another example, consider B = {3,1} and look at how it is represented on a 2-D grid. To add two vectors, place the tail of one (keeping its direction) at the head of the other and connect the first tail to the last head. Finally, subtracting is done just like addition. Just reverse the vector to be subtracted and add as before. (see **A - B** below).

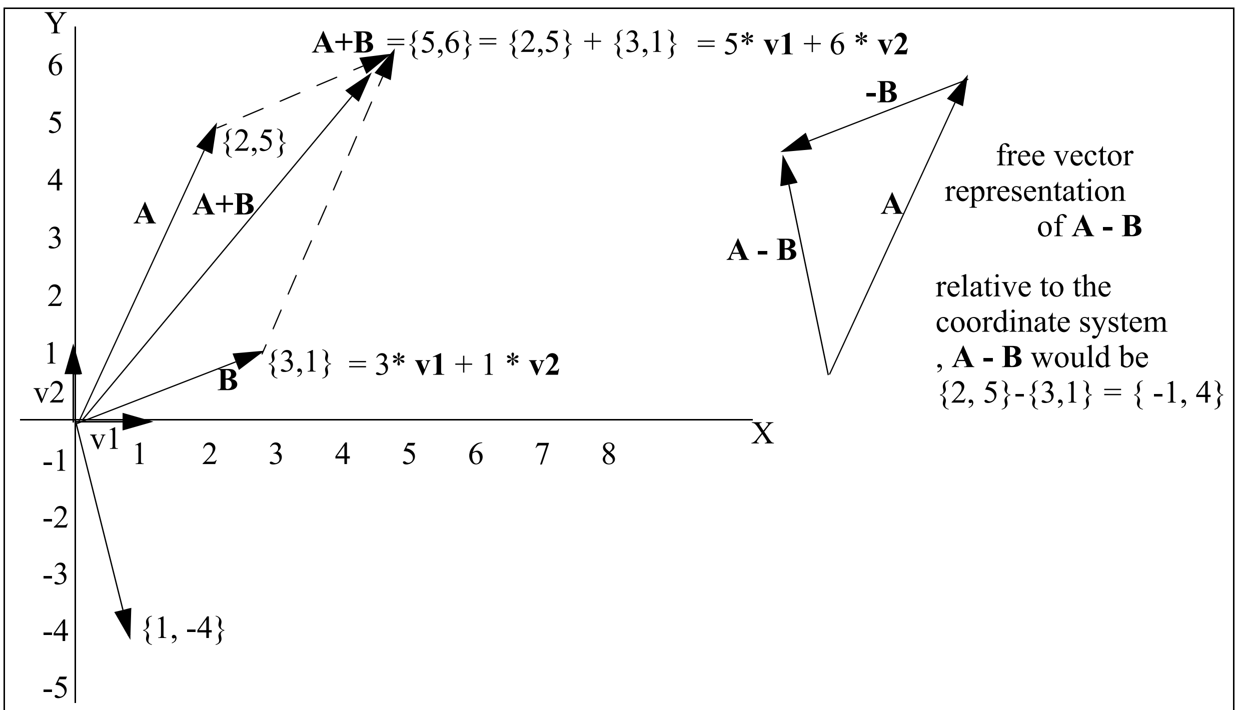


FIGURE 1. A 2-Dimensional Vector Space generated/spanned by basis vectors v1 and v2

These two views, the algebraic and geometric/vector, will constantly interplay, and I, and you, will wind up shifting between interpretations depending on the perspective desired. Notice in the diagram that both interpretations are used (once I embed the vectors in a grid). This 2-dimensional grid is called a 2-Dimensional *Vector Space* since it contains all possible *vectors* with two components.(Similarly, if you have an observation vector with 5 components then you are in a 5-Dimensional Vector Space).

I have also drawn in what are called *basis* vectors, **v1** and **v2** that are one unit long and point along the two dimensions, X and Y. Together, these two special vectors, **v1** and **v2** can represent any vector in the space. That is, a multiple of **v1** plus a multiple of **v2** will get me anywhere in the plane (Vectors Space). For example, I have written **B** as a sum of these two basis vectors, suitably scaled by "3" and "1", just to show that *how any* vector in this space can be written as some combination of **v1** and **v2**. Finally, notice that **A+B** can be displayed either geometrically or written algebraically as {5,6} = 5* v1 + 6* v2. Vectors can be related to a coordinate system or can be un-coordinated! For example, in the physical world vectors can represent entities such as Force or Velocity. These entities exist independent of any coordinate system and are called "free vectors". It is only when I want to calculate numbers that I need a reference coordinate system. The vectors we will deal with in statistics almost always are embedded within a particular vector space, within a given coordinate system.

What's a Projection Length?

A projection length is the part (projection coefficient) of one vector that lies along the same direction as another vector. Look at the diagram above and you will see that the *projection length* of the vector **A+B** down onto vector **v1** is '5'. If you can see that, then you also can see that the length of **A+B** that lies along **v2** is '6' units. This is the divide and conquer approach that allows us to analyze/break-up a problem into components, work on the individual components, and then synthesize an answer back from the pieces. Same thing in statistics. We are going to project a vector (it will turn out to be our vector of *experimental observations*) down onto special unit vectors that are associated with hypothesis test questions. The *resultant projection lengths determine the acceptance or rejection of a specific hypothesis*.

The basis vectors chosen for the above 2-D vector space were particularly easy to work with but won't correspond to any hypothesis of interest. We need to be able to find the projection length along *any* unit basis vector, no matter which way it points. That requires the Dot Product (indicated by a '•'), which I turn to next.

How to do Projections in General - the Utility of the Dot Product.

The first figure, Figure 1 above, used **v1** and **v2** as a basis set for a 2-dimensional vector space. There are many other pairs of vectors that could be used as a basis for this 2-D space though, and I have picked a special pair below, named **u1** and **u2**.(see Figure 2 on page 3). Notice that they are perpendicular to each other as well as being of unit length. (I will verify both these statements in a moment). The task they are suited for is to break up any observation **y** vector by projecting it onto the **u1** vector as well as projecting it onto the **u2** vector. The result is that we have represented **y** as a *sum of two perpendicular vectors*. that are not only easier to work with, but have important statistical meaning. (It will turn out that the length of the projection along **u1** divided by the length of the projection along **u2** is the *t-test,* while the ratio of the *square* of those lengths is the *F-tes*t. Either test lets you check the hypothesis that *y* comes from a distribution with mean *zero*. These two tests are the key tools in all the confirmatory statistical analyses that you will run into..

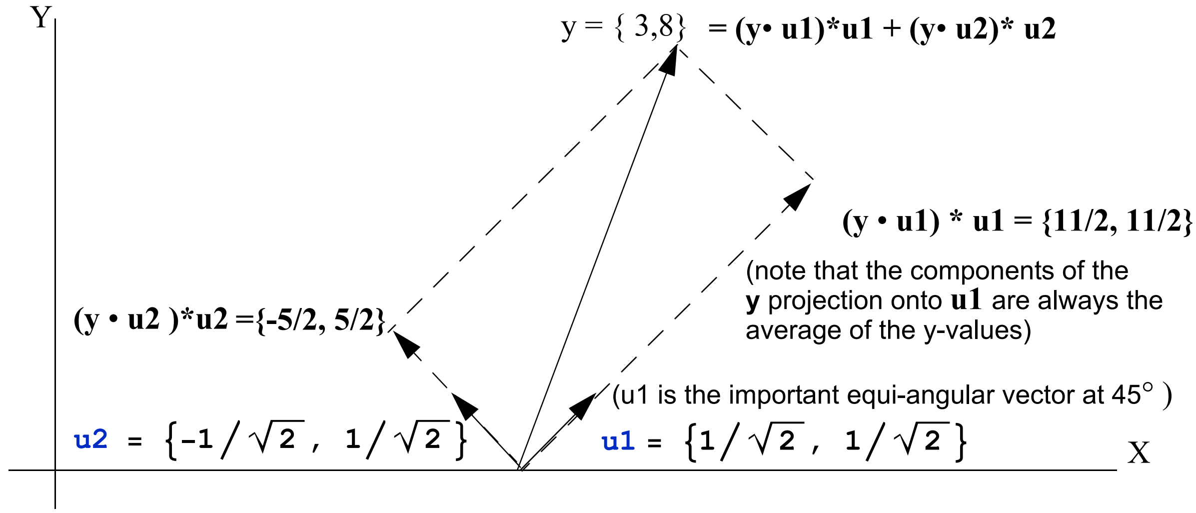


FIGURE 2. Projecting the observation vector 'y' down onto basis vectors u1 and u2

Dot Product Definition and Properties

For any two vectors having general coordinates **A** = {a1, a2, . . . an} and **B** = {b1, b2, . . . , bn} , their Dot Product is written as:

A .• **B**, and is defined as:

|A| * |B| Cos[t], where |A| and |B| are the lengths of the vectors and 't' is the angle, in radians, between them.

Using the coordinates of the two vectors, the Dot Product is also equal to:

A • **B** = a1 * b1 + a2 * b2 + a3 * b3 + . . . an * bn

Dot Product as a Test for Perpendicularity

Since we know that the definition of the Dot Product depends on the cosine of the angle between the two vectors, and we also know that the cosine of 90 degrees is zero, it follows that if the Dot Product is zero, we have vectors at 90 degrees to each other, that is, they are perpendicular. This is an extremely important criterion for perpendicularity, just take the Dot Produce and see if you get zero!

Example 1: let me check out **u1** in relation to **u2.**

length of **u1** = Sqrt [**u1** • **u1**] = 1

length of **u2** = Sqrt [**u2** • **u2**] = 1

u1 • **u2** = -1/2 + 1/2 = 0

Since **u1** • **u2** = |u1| * |u2|* Cos[t], and I know that both **u1** and **u2** have unit length, I can say: Cos[t] must be zero since neither u1 or u2 are,

therefore since Cos[t] = 0, this implies that t = 90°.

Vectors at 90° τo each other are *perpendicular. (* they are also called *orthogonal)*

Example 2: If **y** = { 3,8}, write it in terms of vectors along **u1** and **u2**.

y • **u1** = { 3,8} • { 1,1}/Sqrt[2] = {3+8}/Sqrt[2]

(**y** • **u1**) * **u1** = 11/Sqrt[2] * {1,1}/Sqrt[2] = {11/2, 11/2}

y • u2 = {3,8} • {-1, 1}/Sqrt[2] = 5/Sqrt[2]

(**y • u2**) * **u2 =** 5/Sqrt[2] * { -1, 1}/Sqrt[2] ={ -5/2, 5/2}

Example 3: **A** = {2,5} and **B** = {3,1}

A • B = 2*3 + 5*1 = 11

length of A = Sqrt[**A • A**] = Sqrt[4 + 25] =

length of B = Sqrt[**B • B**] = Sqrt[9 + 1] =

since A • B = |A| * |B| Cos[t] , where 't' is the angle between them, Solving for 't' yields:

Cos[t] = 11/(Sqrt[29]*Sqrt[10]) = 0.65, this is in radian measure

So, the angle between these two vectors is about 50°

Ok, that's about all I want to say in this abbreviated section. I say a lot more below but maybe this much is enough for your purposes?

*** End Cut to the Chase ***

Vectors, their operations, and their enclosures, *vector spaces*, are the essence of statistics in general and multivariate statistics in particular (for that matter, vectors are the essence of physics/engineering as well). Vectors and their spaces allow you to visualize complex interactions in many disciplines that are invisible when using only algebra. This tutorial is a first step to confidently grasping the concepts behind hypotheses testing, degrees of freedom, analysis of variance, (multiple) correlation coefficients, and least squares analyses. This vector material will also be helpful in understanding engineering and physics concepts such as velocity, force, torques, and field descriptions.

I consider this tutorial as a precursor to other tutorials on detailed techniques for performing statistical modeling as in multiple regression, analysis of variance (ANOVA), design of experiments, and the geometric approach to statistical analyses/synthesis in general. If you are pressed for time, just read the next section, *Cut to the Chase*, and then, if still interested, go ahead with the rest of the document. The follow-on tutorials to this one are: *Business Trend Analysis (easy work!)*, *T & F Tests*, and *Multiple Regression Trend Analysis*.

What's a Vector Space and Why Do I Care?

Vector Spaces are the natural habitat of statistics! All of our subsequent tutorials concerning statistical modeling, investigations, and resulting interpretations, will take place within vector spaces. The reason is that vectors and their spaces correspond directly to statistical models and procedures: (this next extract is pretty heavy-duty so don't get discouraged if all is not clear!).

> *Statistical Hypotheses have a direction in space! O*bservation vectors *projected* down into a special *model* subspace enable *different hypothesis tests*.These projection lengths and directions are the bases of the T and F tests. In essence, different lengths and directions in that model space correspond to different hypotheses that can be tested.
>
> *Vector lengths* correspond to internal variability (variance) while *angles* between vectors determine model integrity in general and (multi) *correlation coefficients* in particular.
>
> The major model representations in statistics are: *variable space* representations (scatter-plots) and *subject space* representations (variables as vectors), and both are effected within vector spaces.

Whenever you make a set of observations, fill out a form, or just jot down a number or two, you potentially have a *vector* on your hands, an ordered set of numbers with a direction! Say I listen to a news cast about the early morning winter temperatures around the state of Arizona and I hear: 12 degrees in Flagstaff, 4 at Window Rock, and 9 at the Grand Canyon (South Rim). That's three temperatures I can represent as a set of numbers {12, 4, 9}.

Now though, let me give a geometric interpretation to this set of numbers, the vector interpretation. To do that, I broaden my perspective and think of *all* possible triples of numbers! I go further and set up a space in which all these triples can be embedded - a 3-dimensional 'temperature' space. Finally, I think of each of these possible triples as an arrow starting at the origin {0,0,0} and going out to the triple. Each of those arrows is called a vector anchored at the origin of a 3-dimensional space, *a vector space*, with its tip at the triple point.

Back to freezing temperatures in Arizona - think of the particular temperature vector, {12, 4, 9}, as being one of the potential or possible temperature vectors with 3 components. To get a picture of this vector I set up a 3-dimensional (X-Y-Z) coordinate space and simply draw in this temperature vector. I start at the origin {0,0,0} and draw an arrow out to the point {12, 4, 9}. See "3- Dimensional Vector Space Containing Temperature Triples (with a basis set)" on page 5.

Notice that the geometric perspective introduces a direction (angle) to the ordered set of numbers as well as a length.

In that figure, I have explicitly set up and drawn what is called a basis set, a special set of vectors (u1, u2, u3) pointing along the X-Y-Z axes. A basis set of vectors *generates* the whole space, which in this case is a 3-dimensional box, where *every vector in the space*, such as the Arizona temperature vector {12, 4, 9}, can be represented by some combination of the basis set. When I say *combination,* I am talking about vector operations such as multiplying vectors by real numbers and adding/subtracting multiples of vectors. A very brief discussion of vector operations is next, more detail can be found later in this document.

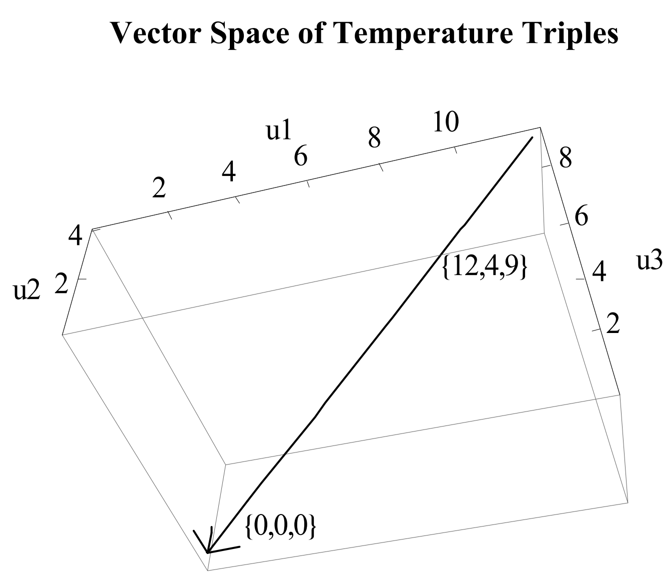


FIGURE 3. 3- Dimensional Vector Space Containing Temperature Triples (with a basis set)

Or, for another vector example, maybe I fill out a vehicle emissions form for my car and list, among other characteristics, these three: {number of cylinders, gross weight, current mileage}. That's a vector too. I could embed this one inside a 3-dimensional space consisting of similar types of vectors and call that vector space, "Car Emissions".

Real Brief Discussion of Vector Operations - Their Geometry & Algebra

For the first vector operation, *scalar multiplication*, just think of stretching or shrinking vectors by using scalar multipliers which change their length but keep (or reverse) their direction. For example, multiplying a vector by '2' doubles its length without changing its direction while multiplying by -2 doubles its length and reverses its direction. To *add* vector Y to vector X, the geometric way is to draw vector X, slide the base of Y to the tip of X (keeping its' direction) and then connect the base of X to the tip of Y. To *subtract* Y from X, multiply Y by -1, and place the base of this vector at the tip of X and then connect base of X to tip of -Y. *This is called the parallelogram law of addition/subtraction.*

As far as the algebra of vectors goes, it's dirt simple - *scalar multiplication* just means multiply each component by the scalar. If X= {12,4,9} and the scalar is '2', then 2* X = {24, 8, 18}

To *add* or *subtract* two vectors, just work it component by component. If X= {12, 4, 9} and Y = {3, -2, 5} then X+Y = {15, 2, 14} and X-Y = {9, 6, 4}, that's it.

These vector operations are importance since they correspond to statistical operations and interpretations. There are two other vector operations I will cover later, Dot products (closely related to vector *Projections*) and Cross products, but, like I said, this is the Cut to the Chase section!

Operations within a Vector Space

I have indicated one choice of this basis set by the mutually perpendicular vectors shown at the origin and labeled them as 'u1', 'u2', and 'u3'. These basis vectors are usually scaled to be of unit length and ideally, are perpendicular to each other. When that happens, the number of basis vectors corresponds to the *dimension* (degrees of freedom) of the vector space and is called an *orthonormal* basis set. In this case I have set up u1 = {1,0,0}, u2={0,1,0}, u3 = {0,0,1} that have the two properties of mutual perpendicularity and unit length. That qualifies them as an orthonormal basis set. Every vector in the space can be represented by a combination of these three. In particular, vectors {1, 1, 1} = 1* u1 + 1 * u2 + 1* u3 and {12, 4, 9} = 12*u1 + 4*u2 +9* u3. (Note: It's a good idea to get a piece of graph paper and draw your own pictures).

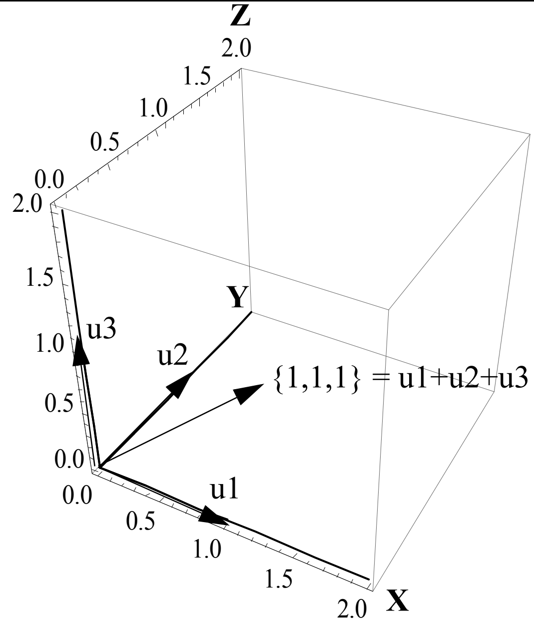


FIGURE 4. \3-Dimensional Vector Space with an Orthonormal Basis

.Using basis vectors to generate a 2-dimensional subspace

Consider all possible combinations of vectors u1, and u2. (Check out the diagram above). All those combinations generate a plane (often called the X-Y plane, an in the floor of a room). Then, any point in the plane, such as {12, 4, 0} for example, is a combination of u1 and u2, namely 12* u1 + 4 * u2. This set of vectors, u1 and u2, generate a 2-dimensional *subspace*. Notice that the third component is always zero and so we don't have that third dimension. Further notice that any combination of vectors in this plane, stay in the plane (that is, what happens in the plane, stays in the plane). This is the crucial feature of a vector subspace, it is closed under vector combinations.

Using basis vector(s) to generate a 1-dimensional space

Consider all the possible multiples of u1. That will generate what we usually think of as the 1-dimensional horizontal or 'X' axis. All multiples of u2 generates the 'Y' axis, while u3 generates the vertical or 'Z' axis. Similarly, all the possible multiples of (u1 + u2 + u3) = {1,1,1} generates the 1-dimensional subspace consisting of the equiangular line through the origin. Again, notice that any vectors in one of these subspaces when combined with another vector in the same subspace, stays in that subspace.

Finally, an arbitrary but fixed combination of u1, u2 and u3 like: 4* u1 -5 *u2 + 8.79*u3 also generates a 1-dimensional subspace, a line through the origin passing through the point {4, -5, 8.79}.

It is a property of vector spaces that I can do this, break up the space into separate non-overlapping subspaces (except for a shared {0,0,0}) within which I can carry out statistical operations independently of what is happening in the other components/subspaces!

> This is a common opportunity/problem-solving approach: break up the task into subtasks hopefully exclusive and exhaustive, which is the *analysis phase*, solve those independent tasks, re-assemble the individual results, which is the *synthesize phase*, learn a little along the way, and repeat as necessary! This is also called the iterative, incremental, concurrent approach.

Vector Operations

Once you have a vector, you can consider it as being part of, or 'embedded' in, a larger, structured collection, called a *Vector Space*. Once you have such a Vector Space and its' constructed and associated vectors, the other features of statistics such as estimates, hypothesis testing, regression, analysis of variance, and inference have a natural setting and can be calculated by Vector Space operations. The *combination* of algebra and geometry represented by a Vector Space allows the computation and insight that will give you guidance in your statistical analyses. You can also find these ideas within the discipline of Linear Algebra which goes deeply into this combination of algebra and geometry.

This tutorial continues on from the 'TrigNotes' tutorial, so you ought to have that one handy. In the Trig tutorial I developed the Pythagorean theorem for a *right* triangle and then went on to demonstrate the law of cosines when the triangle is not *right*. These notes you are now reading introduce vector notation and go on to show arithmetic on vectors in their natural habitat, *Vector Spaces*.

To do statistical analysis we will need to learn about the *Dot Product* of two vectors. The reader will be able to see that the Dot Product is a natural consequence of the law of cosines and so requires no new ideas beyond vector notation. The importance of the Dot Product is that it occurs frequently in all sorts of statistical calculations. For example, we will later learn that the regression correlation coefficient is equal to the Cosine of the angle between two vectors. We get this value by calculating the Dot Product. Also, the idea of a 'projection' of one vector onto another is calculated by means of a Dot Product. Projections will figure in all of the geometric interpretations of the standard tests such as the *t*, *z*, *F*, *ANOVA,* and *Regression*. This tutorial will form part of the basis needed to make these interpretations.

Conventions and Notation:

I will use the following symbols and notations:

1. t - denotes an angle (I will also use *theta*)
2. s - denotes an angle
3. a, b, c, d, e, f, g, h, k, m - denote scalars (regular real numbers such as 4.56 and -3.01)
4. A, B, X, Y- denote vectors, that is, directed line segments generally, but we will specifically use these to represent sample or population observation vectors.See the vector discussion in these notes for more detail.
5. |A|, |B|, |X|, |Y| - denotes the length of these vectors.
6. {a1,a2}, {b1,b2}, {x1,x2}, {y1,y2} - denote the coordinates of a vector once you relate the vector to a particular set of basic vectors forming a coordinate system. The pairs of numbers will be used to show positions in 2-dimensional space. These are the x, y coordinates of the tip of a vector or a general point in space.
7. {a1,a2,a2}, {b1,b2, b3}, {x1,x2,x3}, {y1,y2, y3}- will be used to show positions in 3-dimensional space, that is, x, y, z coordinates.
8. To indicate a variable raised to a power, I will use the symbol" ^". So, if I want to show the square of the variable 'a', I will write a^2. If 'a' is to be cubed, I write a^3 and if I want to show a fractional power like the reciprocal of the square root of 'a', I will write a^(-1/2)

9. A • B indicates the *Dot Product* of two vectors and is equal to the *inner product* of their coordinates. By definition, A • B = |A| * |B| * Cos[theta], where theta is the angle between A and B. Operationally, it consists of multiplying each corresponding coordinate together and then adding those up. The result is a scalar. (this will be illustrated in the body of these notes). For example, if A={1,3}, B= {4, 7}, then A • B = 1*4 + 3*7 = 26. For higher dimensions, the same patterns continues:
   if X = {1,2,3,4, -5}, Y= {5,6,7,8,9} then X• Y = 1*5 + 2*6 + 3*7 + 4*8 -5*9= 25
10. A x B indicates the Cross Product of A with B and is equal to A x B = |A| * |B| * Sin[theta], where theta is the angle between A and B. The result is a vector perpendicular to the plane generated by A and B and in a direction given by the *right hand rule*. This rule says, take your right hand, and rotate vector A into vector B using your fingers. Your thumb points in the direction of A x B. (Note: This definition is from my old physics days, and although there are other ways to indicate the resultant direction, they are pretty abstract!)

Vectors as Packages of Properties

As a first thought about 'vectors' I could say that you use them all day, every day. Whenever you talk about one or more properties of something, and you group these properties together for convenience, you are using the vector idea. A vector simply involves *packaging* up common properties in a standard format. Say I look at my truck parked outside. Among many other properties: it has a weight, horsepower, top speed, color, and load limit. If I like, I could package these properties together into a *vector*, labeled "truck", that consists of:

truck={2000, 125, 95, emerald, 800}

This notation groups these values into a convenient format that can be referred to as a unit. If we pick compatible components, we may be able to do operations on the vectors as a whole rather than consider operations on its individual components. This is a standard pattern in mathematics. If you can think of individual occurrences as components of a higher level object, then that higher level object can be thought about and manipulated on its own terms.

As another common example, when you *fill out a form*, you could think of this as assigning values to a vector that could be distinguished from other forms (vectors) by the entered values. For example, you are often asked to identify your form by including your social security number as one of the entries. Or again, think of a row in a relational database table as a vector of values and the whole table as a set of such vectors.

Our use of vectors will not exploit all of their possibilities, since we will restrict ourselves to only using real numbers as components. That is, our vectors will look like V ={56.4, 3, -21, 98000}, all numbers. Keep in mind though that the concept can accommodate *any* type of entry as a component, even another vector!

Vectors- Free and Bound: Add, Subtract, and Scalar Multiply

Let me take a few minutes to describe the arithmetic we will be doing with vectors. Consider the diagram below, "Vector Arithmetic: Vector Addition, Subtraction, and Multiplication by a Scalar" on page 11. A *scalar* simply means a real number like 38.92 as distinguished from a vector like V = {38.92, 20, 66}. You will notice two perspectives on vectors below. For many physical quantities, like a *Force*, or *Velocity*, they exist independent of what coordinate system you choose to view them in. Vectors that represent these sorts of entities are called *free vectors.* To do calculations, to get numbers though, you will need to reference that vector to a particular coordinate frame.To calculate how many pounds of force are being exerted in a particular direction, you will need to lay out the reference frame you choose to look at the Force from. The same two perspective hold in general. We can talk about vectors without a reference frame (*free vectors*) but when we want to do calculations, we have to choose a particular framework (a particular coordinate system), hence the term *bound vectors*.

Adding Two Vectors

In part (a) I am adding two vectors, A+B, together. Algebraically, this just means add their x-coordinates to get the new x-coordinate and add their y-coordinates together to get the new y-coordinate. Geometrically, it means to take B (keeping its direction), and slide it to the tip of A and connect the origin to this end of the displaced B vector.

Subtracting Two Vectors

In part (b) I subtract A from B, or as written, B-A. There are two perspectives here that you can consider. If I reference A and B to a definite coordinate system with coordinates {a1,a2} and {b1,b2}, then the subtraction operation looks like part {c}.Algebraically, this just means subtract their x-coordinates to get the new x-coordinate and subtract their y-coordinates together to get the new y-coordinate. Geometrically, for these 'bound vectors', take A, slide it to the tip of B and then reverse its direction (the negative of A) and add. The result joins the tip of A to the tip of B, as shown in part (c).

If however, you consider A and B free of a coordinate system (they are actually called 'free vectors' in this case}, then you can indicate their difference by connecting the vector A-B from the tip of A to the tip of B. This must be the correct vector since adding A + (B-A) does get me to B, no matter the coordinate system. That is shown in part (d).

Multiply a Vector by a Scalar

In part (c) I show the B vector multiplied by 1.5. Algebraically, this means that each coordinate of B is multiplied by 1.5. Geometrically, it means lengthen B by a factor of 1.5, keeping its direction constant. Multiplying by a negative number causes the direction to be reversed and then scaled.

Examples:

A={4, 2}, B= {3, 7} where both vectors begin at {0,0}

A+B = {4+3, 2+7} = {7, 9}

B-A = {3-4, 7-2} = {-1, 5}

1.5* B = {1.5* 3, 1.5 * 7} = {4.5, 10.5}

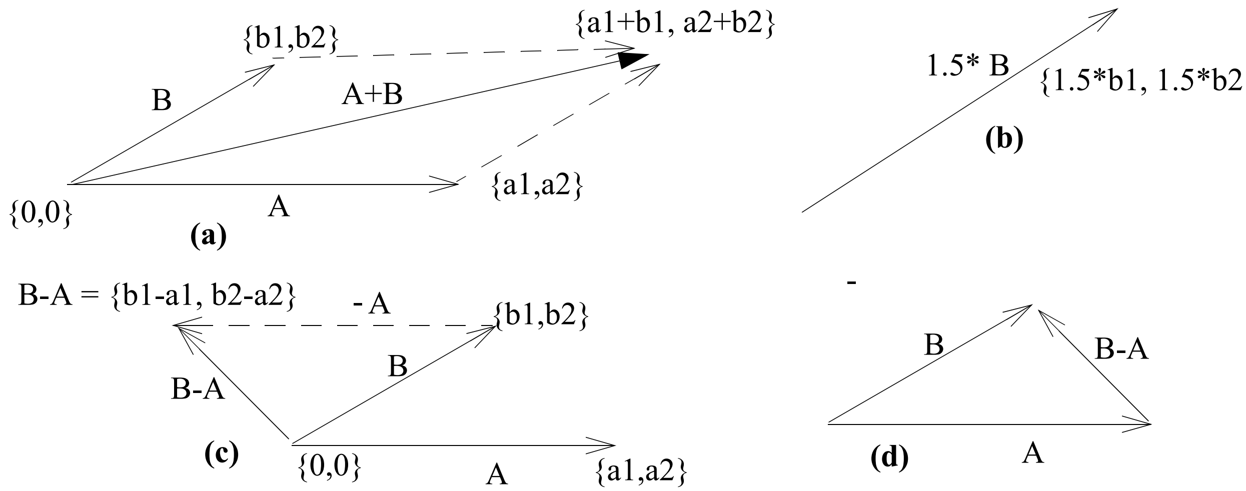


FIGURE 5. Vector Arithmetic: Vector Addition, Subtraction, and Multiplication by a Scalar

Lengths of Vectors and Unit Vectors

Once we have a vector, it's generally useful to know more of its properties. One useful one is to know its length. Let me calculate the lengths of A and B from the previous example. Note that these length calculations use your right triangle knowledge, since every vector length comes from calculating the hypotenuse of a right triangle, if I am using the cartesian coordinate system.

Consider A={4, 2}, B= {3, -7} where both vectors begin at {0,0}

Length of A, denoted as |A| = Sqrt[4^2 + 2^2] = Sqrt[20] = 4.47

To make a unit vector out of A, I simply divide by its length: A/|A| = {0.894, 0.447} If you find the length of this vector, you will get unit length.

For B we have the same approach,

Length of B = |B| = Sqrt[3^2 + -7^2] = 7.6 and making it a unit vector just requires dividing each component by the length, yielding: B/|B| = {3/7.6, 7/7.6} = {0.39, 0.92}

Examples in Three (and higher) Dimensions

For vectors in three dimensions, the procedure is the same (this equates to finding the length of the hypotenuse of a right triangle in three dimensions). Consider

X= {1, 3, 6}, its length would be |X| = Sqrt [1^2 + 3^2 + 6^2] = 6.78

Y= { 4, -2, 3}, |Y| = Sqrt[4^2 + (-2)^2 + 3^2] = 5.38

A= {7,7,7}, its length would be |A| = Sqrt[7^2 + 7^2 + 7^2] = Sqrt[3] Sqrt[49] = 12.12

Applications of the Vector Approach

The Dot Product

The Dot product between two vectors gives a *scalar* measure of their common direction. There are several useful applications of the idea of a Dot product. All of them depend on the fact that the Dot product is a compact way to express the *angle* between two vectors. Formally, the Dot product of two vectors, X and Y, is written as:

X • Y = |X| * |Y| * Cos[theta]

where theta is the angle between the vectors X and Y and |X| and |Y| are the lengths of the vectors.

Basic Trig Origins

1. One perspective on the origin of this idea is found in basic trigonometry operation. It falls out during a derivation of the *Law of Cosines*. (See the section in this tutorial as well as the tutorial *Trigonometry Basics*). Recall that for a triangle with legs X, Y, and side X+Y, the square of the length of side X+Y is:

$|X + Y|^2 = |X|^2 + |Y|^2 - 2\,|X| * |Y| \,\mathrm{Cos[theta]}$

The component of the Law of Cosines, " |X| * |Y| Cos[theta]", *is* the Dot product of X and Y. When theta is 90 degrees, this reduces to the Pythagorean theorem.

Engineering & Physics

1. In engineering/physics, the idea of finding components of physical quantities sharing the same direction appears in numerous equations such as:
   1. **F** • **ds** - the component of force in the direction of the path **s** is a measure of *Work*. The closer the direction of the force vector and path vector, the more work is done. The commonality of direction is measured by the cosine of the angle between the path and the force vector.

Statistics

1. X • Y - for variables X and Y, the Dot product embodies* the Cosine of the angle between the vectors. That cosine value is the (Pearson product moment) Correlation Coefficient, a measure of closeness of the vectors.
2. For an observation vector Y, and a subspace V, spanned by a set of 'p' orthonormal basis vectors, $\{v1, v2, \ldots, v_p\}$, then the component of Y lying in that subspace is the projection of Y onto that space. Projection is another term for Dot product. and would look like:

ProjectionOfYOntoV = Y • v1 + Y • v2 + . . . + Y • v_n

Complex Variables (2-Dimensions only)

In two dimensions (in the complex plane), if I consider the vector X = x1 + I x2, and the vector Y = y1 + I y2, where 'I' is the square root of -1, the Dot product falls out as a consequence of simple complex multiplication, when I multiply X by the Complex Conjugate of Y.

X • Conjugate[Y] = (x1 + I x2) * (y1 - I y2) = x1 y1 + x2 y2 + I (x2 y1 - x1 y2)

The real part of this expression is the *Dot Product,* a scalar, while the imaginary part is what is called the *Cross Product*, a vector. The direction of the Cross Product vector resultant is perpendicular to the plane formed by the two vectors X and Y. If the Cross Product is X x Y, then the resultant direction is given by the 'right hand rule' from engineering.

The Dot Product in Terms of the Law of Cosines (a Review from Trig Basics)

I would like to start using the word 'vector' to stand for 'directed line segment'. The use of vectors is everywhere in engineering, physics, statistics, and the social sciences. It's a good idea to start thinking in those terms even with this beginning tutorial. I will use the word vector in the following two sections freely. In all these field, the Dot product also shows up as a way to understand the relation between vectors representing various physical and statistical/mathematical quantities. For example, in engineering, the amount of work done by a force acting through a displacement is given by the product of the component of that force in the direction of the displacement vector. That component can be expressed by the Dot product of the force with the displacement, written as:

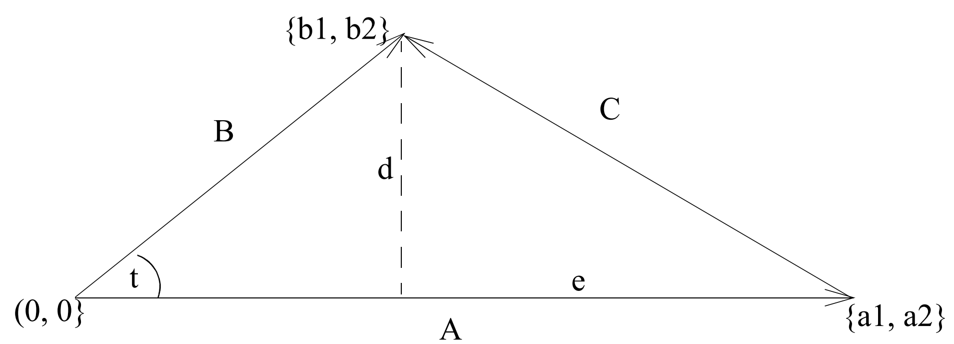
F• d = Work

the Dot product of that force (a vector) with that displacement (a vector) gives the work done by that force (a scalar).

In statistics, the Dot product is used to assess the goodness of a prediction using a predictor. For example If I had a vector representing Income **(I**) and a second representing Education (**E**), how closely are they related? That closeness, called the *correlation coefficient* in statistics, is given by their Dot product **I • E**. Note: technically, the variables have to have their means subtracted off before the Dot product represents their correlation See the tutorial on this site called: *Basic Statistics*)

Now that I have the law of cosines to tell me the length of the directed line segment (vector) 'C' from the above drawing, I can use that to show how the *Dot product* is constructed. Let me now repeat that drawing and add in some coordinates at the tips of vectors A and B, as shown: This means that I have introduced a coordinate system which you can think of as the usual X-Y-Z rectangular coordinate system if you like.

A general triangle used to demonstrate the Dot Product A • B = |A| *|B| Cos[t]



|C|^2 = |A|^2 + |B|^2 - 2* |A| *|B| * Cos[t] by the law of cosines (derived above)

e^2 + d^2 = Pythogoras theorem, right triangle

(a1- b1)^2 + (a2 - b2)^2 =

a1^2 + a2^2 + b1^2 + b2^2 - 2 (a1 b1 + a2 b2) =

|A|^2 + |B|^2

- 2 (a1 b1 + a2 b2) = - 2* |A| *|B| * Cos[t]

a1 b1 + a2 b2 = |A| *|B| * Cos[t] (QED, the Dot Product

FIGURE 6. Demonstration of the Dot Product, A • B = |A| * |B| * Cos[t]

Deriving the Dot and Cross Products using Complex Variables

The interplay between the complex and cartesian representations let me derive the dot and cross product pretty easily. (Thanks to Needham's discussion in his book *Visual Complex Analysis*).

See the diagram below

where s and t are in radian measure and I = Sqrt[-1];

let A = {a1, a2} = |A|(Cos [t]+ i Sin[t]) = |A| Exp[I * t]

let B = { b1, b2} = |B|(Cos[s] + i Sin[s]) =|B| Exp[I * s]

Bconjugate = {b1, -b2} = |B| Exp[I * (-s)]

If I multiple A by the conjugate of B the resultant real part gives both the dot product while the imaginary part gives the cross product.

A * Bconjugate =|A|*|B| * Exp [I * (t-s)]

writing A as a1 + I * a2 and writing the conjugate of B as b1 - I b2 and multiplying:

(a1 + I a2) * (b1 -I b2) = a1 b1 + a2 b2 + I (a1 b2 - b1 a2)

on the right hand side I have

|A|*|B| (Cos[t-s] + I Sin[t-s])

|A|*|B| Cos[[t-s] +I |A|*|B| Sin[t-s]

equating real and imaginary parts I have:

a1 b1 + a2 b2 = |A|*|B| Cos[t-s] which is the dot product of those two vectors

a1 b2 - a2 b1 = |A|*|B| Sin[t-s] which is the cross product. (except for a perpendicular to the plane formed by A and B unit direction vector, (called a *normal* direction) that can be determined by the order of multiplication).

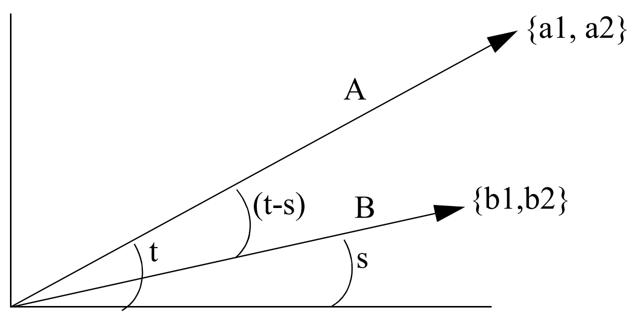
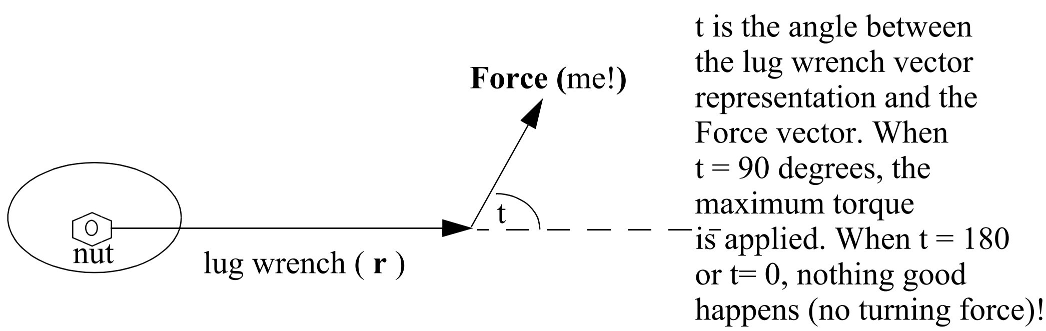


FIGURE 7. Deriving the Dot and Cross Product using Complex Variables

Physical Realization of the Cross Product

Another operation between vectors is of considerable importance, the Cross product. This operation arises most clearly in engineering and physics calculations where the result of the vector operations is another vector which is perpendicular to both of the operand vectors. This needs an example! In engineering there is a quantity called *torque*, which is a measure of how hard it is to get an object rotating. You are familiar with this task when you try to loosen a wheel nut to fix a flat tire. See the diagram below. To remove the nut you use a lug wrench with a long handle. When you apply a force at *right* angles to the wrench handle, the nut turns (hopefully!) Your force, applied at the end of the 'lever' constitutes a *torque*. If you apply your force at right angles to the handle, you will apply more torque. As you adjust your angle of 'push' away from 90 degrees relative to the handle, you diminish your torque and the applied force to the nut. In the extreme case, if you pushed/pulled along the direction of the lug wrench, nothing good will happen (that is, no turning force). The determining factor here is the *sine* of the angle between your push and the direction of the lug wrench handle. 90 degrees does the best job, which means Sin[90] = 1. Here is a picture that will help!



Cross Product **r x F** = | r | * |F |* Sin[t] **n (**where **n** is a unit vector out of the page, toward you)

The formal definition the cross product of the two vectors wrench, (**r**) and Force, (**F**) is:

torque = **r x F** = |r| * |**F**| * Sin[t] **n**

where 't' is the angle between **r** and **F**. The resultant vector, called *torque*, is a vector perpendicular to both **F** and **r** with magnitude |F| *| r | * Sin[t] and direction **n**. That direction **n** is given by the right hand rule which says: put your fingers along **r**, and rotate your hand into **F**, your thumb then points in the resultant direction, **n**. Think of sliding **r** along the dotted line until its base is at the base of F, and then rotate it into F. The result will be a vector of magnitude |F| *| r | * Sin[t], pointing out of the page.

The Cross Product as the Area of a Parallelogram

In geometry you have to calculate all kinds of lengths and areas and one of these kinds of areas is called a parallelogram. This is a four sided figure with two parallel sides having the same length. Look at the next figure and notice that the parallelogram is built from two vectors, A and B

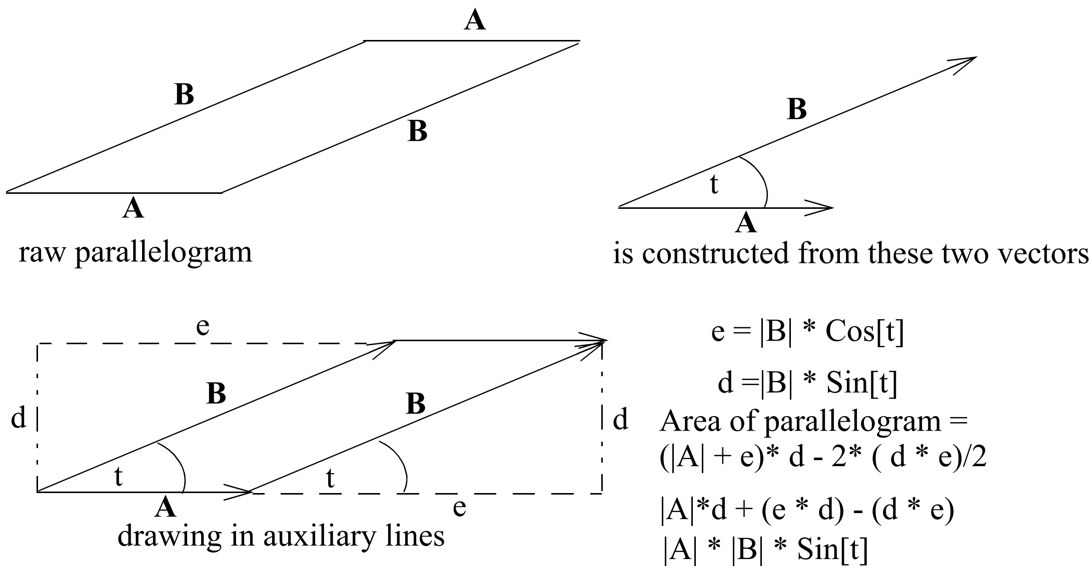


FIGURE 8. Area of a Parallelogram = Cross Product of Two Vectors

From the diagram above you can see how to calculate the area of a parallelogram and by noting an earlier definition of the Cross product of two vectors, you can see that the area of the parallelogram is the magnitude of the Cross product of the vectors used in the construction of the parallelogram.

Projections of One Vector Onto Another - Applying the Dot Product

In statistics, we will *project* the observation vector down onto *basis* vectors that characterize various subspaces of the overall Vector Space we are working in. A distinguished subspace we will always be dealing with is a subspace called the *Model Space*. I will talk more about the Model Space in the "Normal Family" tutorial but for now just think of it as the space of long-run outcome vectors, that is, all the expected value vectors.These long run outcome vectors are the ones we are doing statistics tests on. We do a projection down into this space to see to what extent the outcome of our *experiment* supports our hypotheses or conjectures involving some of these long run outcomes. Each allowable hypothesis or conjecture, is represented by a *direction* down in the Model Space and the *length* (or the square of the length) of the projection of the observation vector down along this direction gives clues as to the truth of the hypothesis.

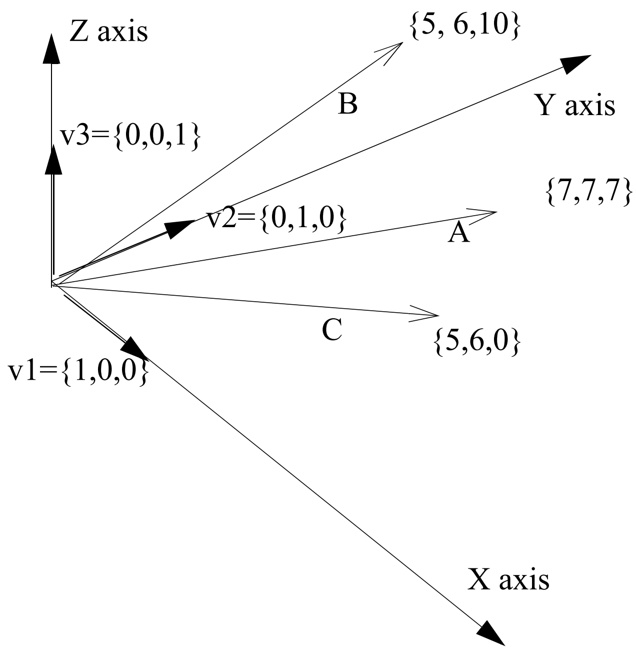


FIGURE 9. The Cartesian Coordinate System with Unit Orthogonal Vectors, v1, v2, v3

From the example above, notice that the *projection* of B down onto the v1 basis vector is the *vector* B • v1 * v1 = 5* v1 = {5,0,0}. Similarly the resultant projection vector of B down onto v2 and v3:

B• v2 * v2 = 6* v2 = {0,6,0} and

B • v3 * v3 = 10* v3 = {0,0,10}

Notice that the length of each projection vector is the Dot Product, given that the projection vector is along a basis vector. For example, the *length* of the projection of B down onto the v1 unit length basis vector is:

v • v1 = 5

Similarly, the lengths of the projection vectors down on v2 and v3 are '6' and '10' respectively.

Demonstration of the Law of Cosines

The plan here is to describe the length of side C, of a *general triangle*, in terms of the lengths of A, B and the angle 't' between A and B. It turns out that it is easier to first describe how the *square* of C relates to the *squares* of A and B. That is, how |C|^2 relates to |A|^2 and |B|^2. The reason for this is that we will be able to use the Pythagorean theorem as a component of the demonstration.

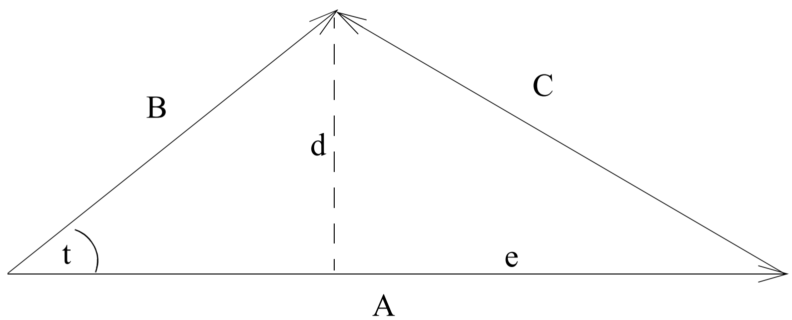
See if you agree with the steps I will show next:

1. drop a perpendicular from the tip of B to A. That is the line whose length is denoted as 'd' and make a right angle with A. Notice that this breaks up the original triangle into two *right* triangles.That is real progress since we have recast the original question by constructing two more questions, but which are simpler. That is, we have broken up a general triangle into two special ones (right triangles) that we know how to work with.
2. the line segment marked 'e' is the length from the tip of A to the base of 'd'.
3. |C|^2 = d^2 + e^2 by Pythogoras (and now you know how to prove that!).

4. d = |B| Sin[t] from the diagram, referring to your trig function definition of Sin[t]
5. e = |A| - |B| Cos[t]
6. |C|^2 = d^2 + e^2 = |B|^2 Sin[t]^2 + |A|^2 + |B|^2 Cos[t]^2 - 2*|A| *|B| * Cos[t]
7. remember that sine squared plus cosine squared is ‘1’, So, factor out the |B|^2 term from the sine and cosine squares and you will get:
8. |C|^2= d^2 + e^2 = |A|^2 + |B|^2 {Sin[t]^2 + Cos[t]^2} -2*|A| * |B| * Cos[t]
9. |C|^2 = |A|^2 +|B|^2 - 2* |A| * |B|

QED.(quod erat demonstrandom -that which was to be proved)

A general triangle used to demonstrate the law of cosines



|C|^2 = d^2 + e^2 = (|B|Sin[t])^2 + (|A| - |B|Cos[t])^2

= |B|^2 {Sin[t]^2 + Cos[t]^2} + |A|^2 - 2* |A|* |B|* Cos [t]

= |A|^2 + |B|^2 - 2* |A| *|B| Cos[t]

FIGURE 10. Demo of the Law of Cosines (directed line segments, A, B, and C as *vectors)*

The Dot Product in Terms of the Law of Cosines

I would like to start using the word ‘vector’ to stand for ‘directed line segment’. The use of vectors is everywhere in engineering, physics, statistics, and the social sciences. It’s a good idea to start thinking in those terms even with this beginning tutorial. I will use the word vector in the following two sections freely. In all these field, the Dot product also shows up as a way to understand the relation between vectors representing various physical and statistical/mathematical quantities. For example, in engineering, the amount of work done by a force acting through a displacement is given by the product of the component of that force in the direction of the displacement vector. That component can be expressed by the Dot product of the force with the displacement, written as:

F• d = Work

the Dot product of that force (a vector) with that displacement (a vector) gives the work done by that force (a scalar).

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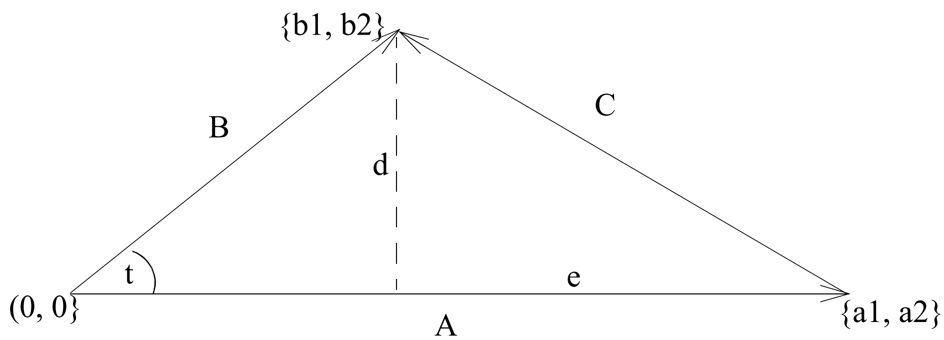
Now that I have the law of cosines to tell me the length of the directed line segment (vector) 'C' from the above drawing, I can use that to show how the *Dot product* is constructed. Let me now repeat that drawing and add in some coordinates at the tips of vectors A and B, as shown: This means that I have introduced a coordinate system which you can think of as the usual X-Y rectangular coordinate system if you like. The conclusion, derived below, we will come to is that: for two vectors **A** and **B**, the Dot Product looks like:

A • B = |**A**|* |**B**| * Cos[angle between them], ; where |**A**| and |**B**| denote the lengths of **A** and **B**

A general triangle used to demonstrate the Dot Product A • B = |A| *|B| Cos[t]



|C|^2 = |A|^2 + |B|^2 - 2* |A| *|B| * Cos[t] by the law of cosines (derived above)

e^2 + d^2 = Pythogoras theorem, right triangle

(a1- b1)^2 + (a2 - b2)^2 =

a1^2 + a2^2 + b1^2 + b2^2 - 2 (a1 b1 + a2 b2) =

|A|^2 + |B|^2

- 2 (a1 b1 + a2 b2) = - 2* |A| *|B| * Cos[t]

a1 b1 + a2 b2 = |A| *|B| * Cos[t] QED, the Dot Product

FIGURE 11. Demonstration of the Dot Product, A • B = |A| * |B| * Cos[t]

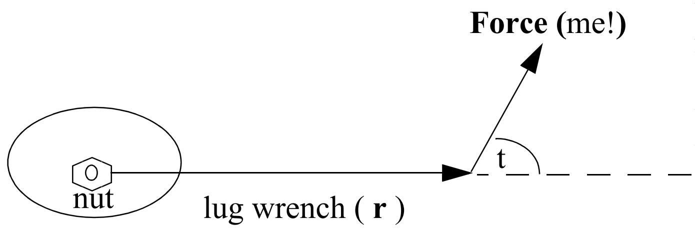
Dot Product as a Test for Perpendicularity

Since we know that the definition of the Dot Product depends on the cosine of the angle between two vectors, and we also know that the cosine of 90 degrees is zero, it follows that if the Dot Product is zero, we have vectors at 90 degrees to each other, that is, they are perpendicular. This is an extremely important criterion for perpendicularity, just take the Dot Produce and see if you get zero!

Cross Product in Terms of Torque

Another operation between vectors is of considerable importance, the Cross product. There is a trig interpretation of this operation which is why I have included it here. Note: I cover this more thoroughly in another tutorial on this site: *Vector Arithmetic & Vector Operations.*

This operations arises most clearly in engineering and physics calculations where the result of the vector operations is another vector which is perpendicular to both of the operand vectors. This needs an example! In engineering there is a quantity called *torque*, which is a measure of how hard it is to get an object rotating. You are familiar with this task when you try to loosen a wheel nut to fix a flat tire. See the diagram below. To remove the nut you use a lug wrench with a long handle. When you apply a force at *right* angles to the wrench handle, the nut turns (hopefully!) Your force, applied at the end of the ‘lever’ constitutes a *torque*. If you apply your force at right angles to the handle, you will apply more torque. As you adjust your angle of ‘push’ away from 90 degrees, you diminish your torque and the applied force to the nut. In the extreme case, if you pushed/pulled along the direction of the lug wrench, nothing good will happen. The determining factor here is the *sine* of the angle between your push and the direction of the lug wrench handle. 90 degrees does the best job, which means Sin[90] = 1. Here is a picture that will help!



t is the angle between the lug wrench vector representation and the Force vector. When t = 90 degrees, the maximum torque is applied. When t = 180 or t= 0, nothing good happens!

Cross Product **r x F** = | r | * |F |* Sin[t] **n** (where **n** is a unit vector out of the page, toward you)

The formal definition the cross product of the two vectors wrench, (**r**) and Force, (**F**) is:

torque = **r x F** = |r| * |**F**| * Sin[t] **n**

where ‘t’ is the angle between **r** and **F**. The resultant vector, called *torque*, is a vector perpendicular to both **F** and **r** with magnitude |F| *| r | * Sin[t] and direction **n**. That direction **n** is given by the right hand rule which says: put your fingers along **r**, and rotate your hand into **F**, your thumb then points in the resultant direction, **n**. Think of sliding **r** along the dotted line until its base is at the base of F, and then rotate it into F. The result will be a vector of magnitude |F| *| r | * Sin[t], pointing out of the page.

Cross Product in Terms of Area of a Parallelogram

In geometry you have to calculate all kinds of lengths and areas and one of these kinds of areas is called a parallelogram. This is a four sided figure with two parallel sides having the same length. Look at the next figure and notice that the parallelogram is built from two vectors, A and B (you may also note that it is built from two triangles).

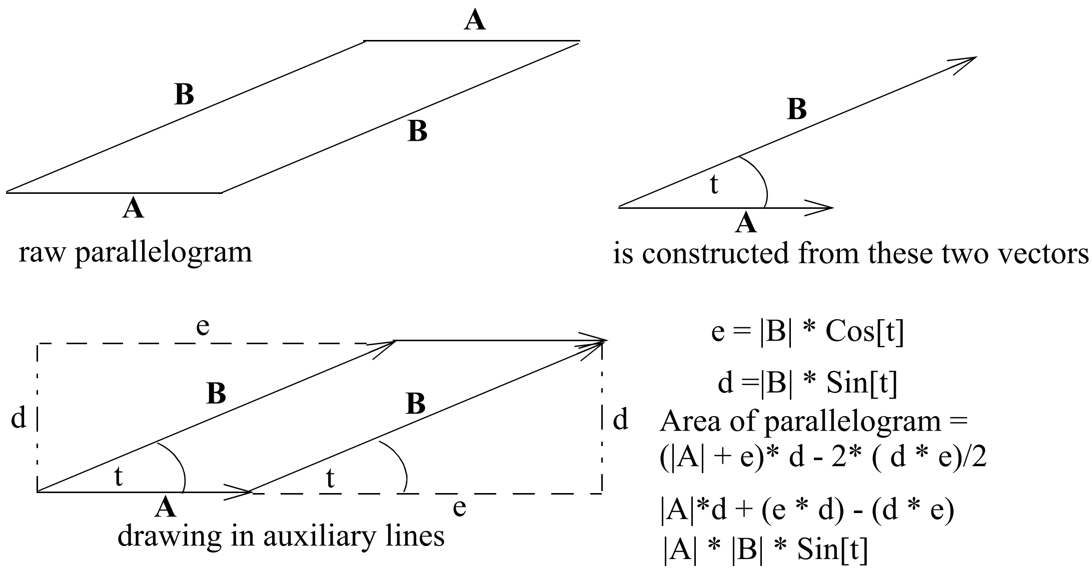


FIGURE 12. Area of a Parallelogram = Cross Product of Two Vectors

From the diagram above you can see how to calculate the area of a parallelogram and by noting an earlier definition of the Cross product of two vectors, you can see that the area of the parallelogram is the magnitude of the Cross product of the vectors used in the construction of the parallelogram.

*As a bonus, notice that the area of the triangle formed by A and B is just half of the parallelogram. So the area of a triangle with included angle is:

Area Triangle = |A| * |B| * Sin[t] /2

Finally, notice that the parallelogram is just the same triangle, joined together twice.

Summary

Now we have a powerful set of tools: Trig functions, Pythogoras' theorem, the Law of Cosines, and the Dot Product as well as Vector Spaces. With these, we can calculate the lengths of sides of arbitrary triangles, solve for the Cosine (or any trig function) of the angle between two vectors (and so find the angle), and find the projection of one vector onto another. Having a Vector Space with a basis chosen with respect to the hypotheses we are interested in, will give us a way to interpret the outcomes of the important statistical tests. These ideas are explored further in follow-on tutorials such as:

- Business Trend Analysis
- T&F Tests
- Multiple Regression Trend Analysis

Extra Verbiage & 1/4 Baked Thoughts

2008-09-04 thinking of introducing the wickens S and S' subspace as basic ideas... ---

Playing with Wickens, and physics. Reference frames S- the whole space, and S' ref frame (the constant subspace). Was thinking that S is the N- dimensional observation space, while S' is the constant space of x1bar, x2bar and such. So the centered variables x1, x2 ,,, , are relative to the S' refrences frame. So, the equiangular line is the S' reference frame and is the {1,1,1,1,1,} subspace Break up any vector into its S' (const portion and its Perp vector (centered vector) How show this easiest hmmmmmmmm

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