Vector Operations Quick Look (*Draft 2009-08-14)

The intent of this tutorial is to provide minimal background for working with vectors and vector spaces, with a view to ultimately understanding statistical procedures. I first describe the basic units of statistical analysis, *vectors*, and then show how to manipulate them. A key operation on pairs of vectors, expressing their angular separation, is called the *dot* product. I will cover the dot product in this tutorial. (Note: the *cosine* of that angular separation is the correlation coefficient whose interpretation using vectors is very clear, while the algebraic description is very unclear). Vectors associated with a particular domain of analysis can be considered to be embedded inside of what is called a *vector-space.* Vector spaces allow statistical (and engineering and physics for that matter) operations on their vector members that yield insight and solutions into all types of questions. Again, the dot product is a key operation that is used to break-up experimental data vectors into perpendicular sub-spaces that have important statistical meaning, meanings that you can actually see from the geometry. Along these lines, check out *BasicStatsBivariateRegression.pdf* on this web site for examples of statistical visualization.

This is a very short excerpt from a longer document *Vector Operations and Vector Spaces,* that goes into much more detail about these topics. A background tutorial called *Trigonometry Basics* might also be helpful if you are a bit hazy about Pythagoras' theorem, or trig functions.

Cut to the Chase (for those with no time!)

This "Cut to the Chase" section of the tutorial is mercifully brief. I intend this to be background to later tutorials needing vector operations and vector spaces to set up and analyze statistical/quantitative experiments. The follow-on tutorial that relies on this one is *GeoStatistics Part I* on the *milagrosoft.com* site. Check out the two diagrams shown next for better understanding.

A Vector == list of values (algebraic view), directed line segment (geometric view)

The plan is to turn a list of numbers into a graphic, called a 'vector'. In the diagram below, **A** = {2, 5} is a list of two numbers, 2 and 5, in that order. That's the algebraic perspective. I can also 'place' that list on a 2-dimensional grid and interpret it as a "vector" line segment with **A** starting at {0,0} and going out to {2,5}. That's the picture, the *geometric view.* More colorfully, the *tail* of **A** is at {0,0} and its *head* is at {2,5}. For another example, consider B = {3,1} and look at how it is represented on a 2-D grid. To add two vectors, place the tail of one (keeping its direction) at the head of the other and connect the first tail to the last head. Finally, subtracting is done just like addition. Just reverse the vector to be subtracted and add as before. (see **A - B** below). The vectors we can deal with in statistics and engineering can be of three types. *Free vectors*, line vectors and bound vectors. *I will work with 'free vectors' since those are the ones that can be moved about the plane (or space) so long as their magnitude and direction are kept the same.* When I want to calculate though, then I will place these vectors inside a particular coordinate system. In fact, whenever I give numbers for a vector's components, I have already implicitly chosen a coordinate system.

The operations of adding two vectors together is sometimes called the "triangle law" of vector addition or the parallelogram law of vector addition. I hope you can see that from the diagrams below. If not, get some graph paper and experiment with various configurations until these rules seem plausible. Its a little tricky to see that the vectors exist apart from any grid or coordinate system but once embedded inside a grid with a concrete set of bases, then numbers can be calculated. That is shown in the left part, **(a)**, while **(b)** and **(c)** treat the vectors as free.

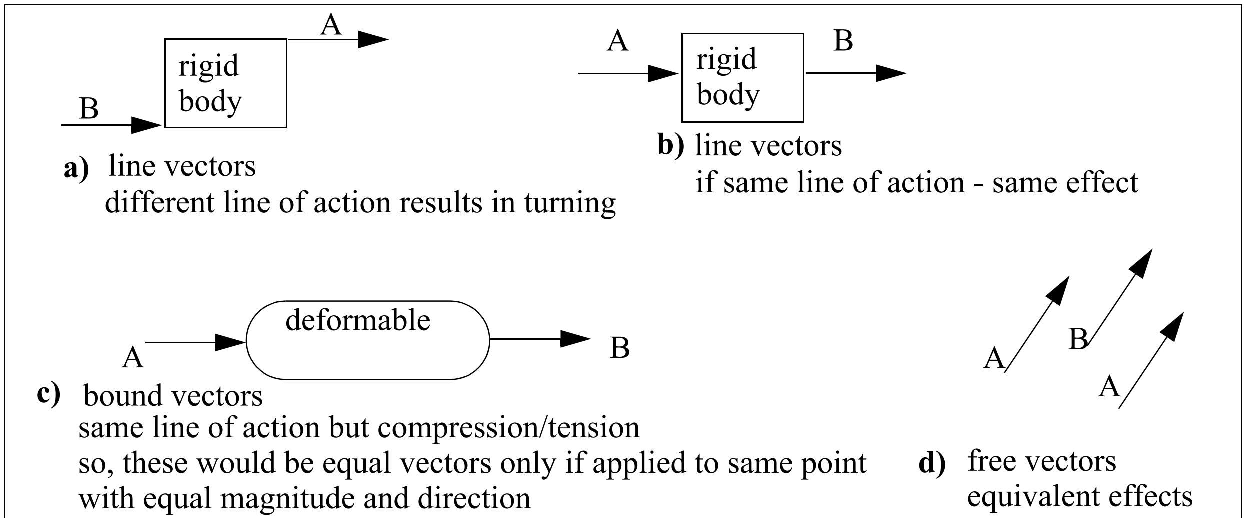

FIGURE 1. Free vectors can be freely moved if size and direction are kept. Line and bound, not

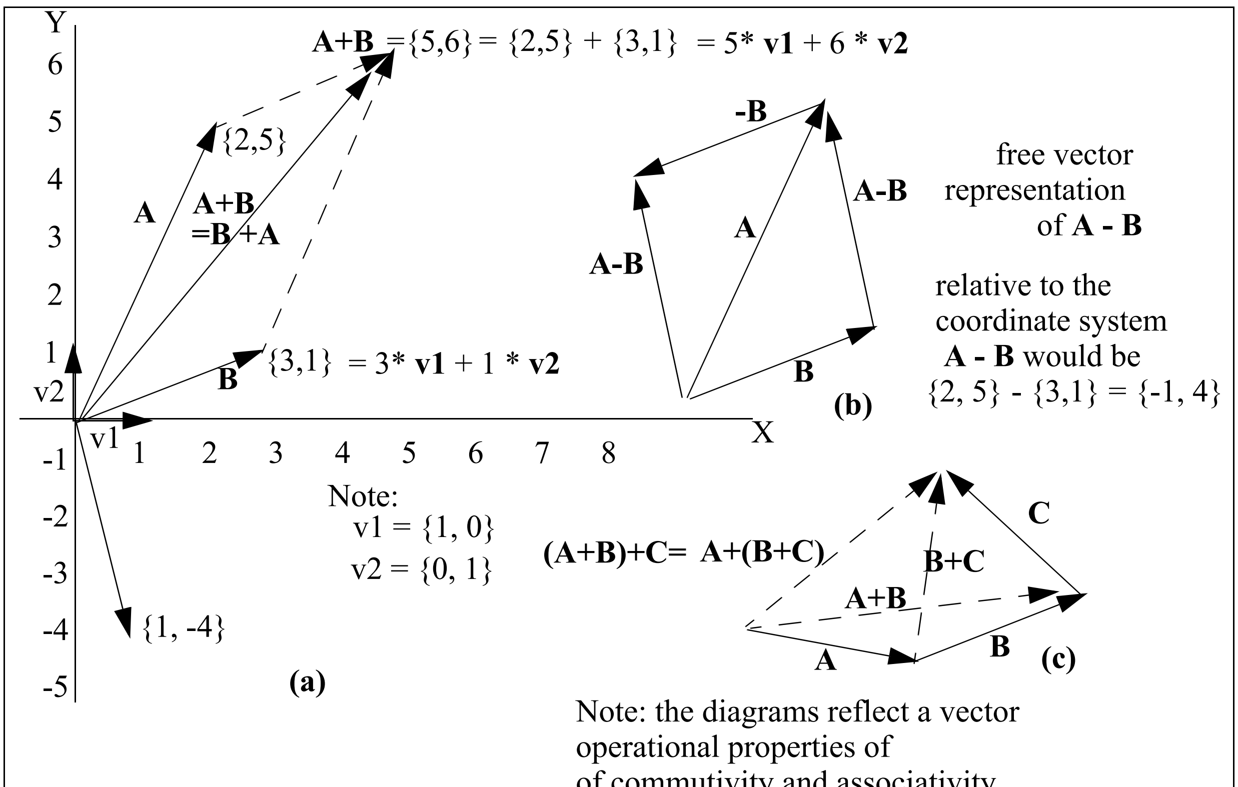

FIGURE 2. A 2-Dimensional Vector Space generated/spanned by basis vectors v1 and v2

Vector Space == collection of compatible component vectors, with operations

These two views, the algebraic and geometric/vector, will constantly interplay, and I, and you, will wind up shifting between interpretations depending on the perspective desired. Notice in the diagram above that both interpretations are used (once I embed the vectors in a grid). This 2-dimensional grid is called a 2-Dimensional *Vector Space* since it contains all possible *vectors* with two components. (Similarly, if you have an observation vector with 5 components then you are in a 5-dimensional Vector Space). A Vector Space is then a set of vectors, all with the same number of components, that supports the operations of vector addition and scalar multiplication. These are the operations you are seeing in the diagrams. (For the Math wizards, the vectors form a group under the *vector* addition operations, the scalars form a *field* with their two operations of addition and multiplication, and finally, the scalar field and vector group are connected by the operation that allows the scalar to multiply the vector, thus resulting in a change of direction or length of the vector. This whole set of objects and operations comprise a vector space).

A Vector Space has a basis (actually, many possible bases)

I have also drawn in what are called *basis* vectors, **v1** and **v2** that are one unit long and point along the two axes, X and Y. Together, these two special vectors, **v1** and **v2** can represent any vector in the space. They are said to 'span' the space in the sense that all possible combinations of **v1** and **v2** generate the whole space. That is, a multiple of **v1** plus a multiple of **v2** will get me anywhere in the plane (Vectors Space). For example, I have written **B** as a sum of these two basis vectors, suitably scaled by "3" and "1", just to show that how any vector in this space can be written as some combination of **v1** and **v2**. Finally, notice that **A+B** can be displayed either geometrically or written algebraically as {5,6} = 5* v1 + 6* v2 = {5,0} + {0,6}

Vectors can be related to a coordinate system or can be un-coordinated! For example, in the physical world vectors can represent entities such as Force or Velocity. These entities exist independent of any coordinate system and are called "free vectors". It is only when I want to calculate numbers that I need a reference coordinate system. The vectors we will deal with in statistics almost always are *free* vectors, embedded within a particular vector space, within a given coordinate system.

What's a projection length?

A projection length is the part (projection coefficient) of one vector that lies along the same direction as another vector. Look at the diagram above and you will see that the *projection length* of the vector **A+B** down onto vector **v1** is '5'. If you can see that, then you also can see that the length of **A+B** that lies along **v2** is '6' units. This is the divide and conquer approach that allows us to analyze/break-up a problem into components, work on the individual components, and then synthesize an answer back from the pieces. Same thing in statistics. We are going to project a vector (it will turn out to be our vector of *experimental observations*) down onto special unit vectors that are associated with hypothesis test questions. The *resultant projection lengths determine the acceptance or rejection of a specific hypothesis*.

The basis vectors chosen for the above 2-D vector space were particularly easy to work with but won't correspond to any hypothesis of interest. We need to be able to find the projection length along *any* unit basis vector, no matter which way it points. That requires the Dot Product (indicated by a '•'), which I turn to next.

How to do projections in general - the utility of the Dot Product.

The first figure, Figure 1 above, used **v1** and **v2** as a basis set for a 2-dimensional vector space. These vectors were particularly easy to work with and find projections. There are many other pairs of vectors that could be used as a basis for this 2-D space though, and I have picked a special pair below, named **u1** and **u2**.(see Figure 3 on page 4). The vector **u1** is particularly important in statistics as it carries the projection of the mean of the observation vector. (The length of this projection vector will be used to test for the true mean of a distribution).

Notice that they are perpendicular to each other as well as being of unit length. (I will verify both these statements in a moment). The task they are suited for is to break up any observation **y** vector by projecting it onto the **u1** vector as well as projecting it onto the **u2** vector. The result is that we have represented **y** as a *sum of two perpendicular vectors*. that are not only easier to work with, but have important statistical meaning. (It will turn out that the length of the projection along **u1** divided by the length of the projection along **u2** is the *t-test,* while the ratio of the *square* of those lengths is the *F-tes*t. Either test lets you check the hypothesis that *y* comes from a distribution with mean *zero*. These two tests are the key tools in all the confirmatory statistical analyses that you will run into.

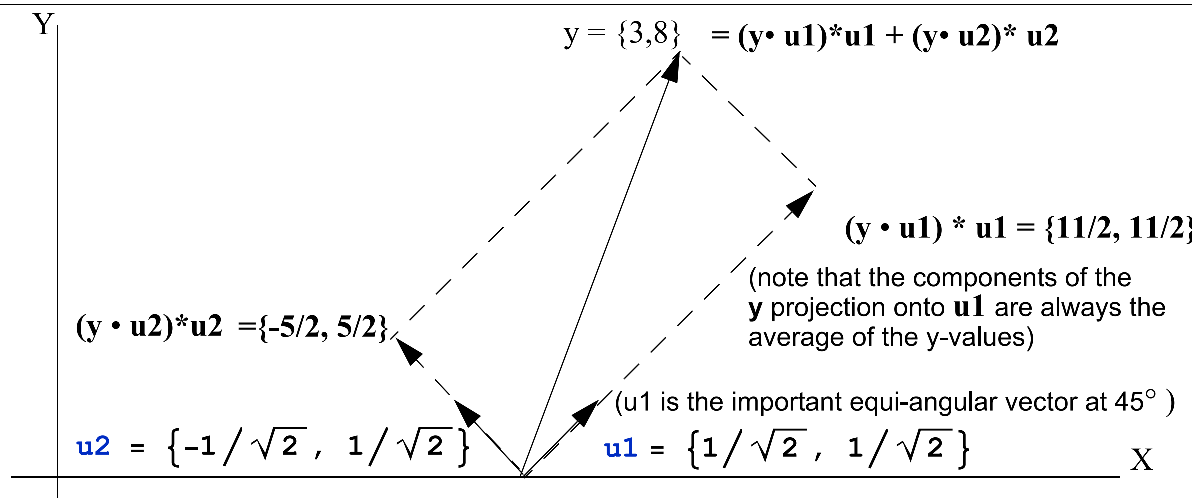

FIGURE 3. Projecting the observation vector ‘y’ down onto basis vectors u1 and u2

Dot Product Definition and Properties

For any two vectors having general coordinates **A** = {a1, a2, . . . an} and **B** = {b1, b2, . . . , bn}, their Dot Product is written as:

A • **B**, and is defined as:

|A| * |B| Cos[t], where |A| and |B| are the lengths of the vectors and ‘t’ is the angle, in radians, between them.

Using the coordinates of the two vectors, the Dot Product is also equal to:

A • **B** = a1 * b1 + a2 * b2 + a3 * b3 + . . . an * bn

Dot Product as a Test for Perpendicularity (also called Orthogonality)

Since we know that the definition of the Dot Product depends on the cosine of the angle between the two vectors, and we also know that the cosine of 90 degrees is zero, it follows that if the Dot Product is zero, we have vectors at 90 degrees to each other, that is, they are perpendicular. This is an extremely important criterion for perpendicularity, just take the Dot Produce and see if you get zero! This property is also used to put constraints on an equation as well, that is, if the dot product is zero, what does that say about the parameters of the equation?

Example 1: let me check out **u1** in relation to **u2.**

length of **u1** = Sqrt [**u1** • **u1**] = 1 length of **u2** = Sqrt [**u2 • u2**] = 1

u1 • u2 = -1/2 + 1/2 = 0

Since **u1 • u2** = |u1| * |u2|* Cos[t], and I know that both **u1** and **u2** have unit length, I can say: Cos[t] must be zero since neither u1 or u2 are,

therefore since Cos[t] = 0, this implies that t = 90°.

Vectors at 90° τo each other are *perpendicular. (*they are also called *orthogonal)*

Example 2: If **y** = {3,8}, write it in terms of vectors along **u1** and **u2**.

y • u1 = {3,8} • {1,1}/Sqrt[2] = 11/Sqrt[2]

(**y • u1**) * **u1** = 11/Sqrt[2] * {1,1}/Sqrt[2] = {11/2, 11/2}

y • u2 = {3,8} • {-1, 1}/Sqrt[2] = 5/Sqrt[2]

(**y • u2**) * **u2 =** 5/Sqrt[2] * {-1, 1}/Sqrt[2] ={-5/2, 5/2}

Example 3: **A** = {2,5} and **B** = {3,1}

A • B = 2*3 + 5*1 = 11

length of A = Sqrt[**A • A**] = Sqrt[4 + 25] =

length of B = Sqrt[**B • B**] = Sqrt[9 + 1] =

Finding the angle between two vectors

since A • B = |A| * |B| Cos[t], where 't' is the angle between them, Solving for 't' yields:

Cos[t] = 11/(Sqrt[29]*Sqrt[10]) = 0.65, this is in radian measure

Now I need to find the angle whose Cosine is 0.65. The ArcCos (the angle whose Cosine is. . .) function is what I need, and applying it gives me the angle whose Cosine is 0.65.

That angle 't' between these two vectors, is very close to 50°

Ok, that's about all I want to say in this abbreviated section. I say a lot more in the tutorial *Vector Operations and Vector Spaces*, but I hope this much is enough for your purposes.

Summary

This 'Quick Look' is an excerpt from the longer tutorial *Vector Operations and Vector Spaces* and I would suggest you look at that one if you want more depth and examples. The bottom line, so to speak, is that vector representations allow another step of abstraction that makes problem solving easier. Vectors let you treat a whole group of numbers as a unit, and even better, let you examine that unit within a richer environment called a *vector-space*. All of statistics relies on vectors for compact descriptions and vector-matrix solutions. Finally, vectors, as geometric objects representing *variables*, let us human beings use our highly developed *visual intuitions* to understand and to guess at interrelationships that can then be verified by further computer and algebraic computations.